Toward Realistic Integrable Gauge Theories and Conformal Gravity in Twistor Strings

Jay N. Ihry

A dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Physics and Astronomy.

Chapel Hill
2009

Approved by:

Louise Dolan

Charles Evans

Jack Ng

Ronen Plesser

Katrin Wendland

Abstract

JAY N. IHRY: Toward Realistic Integrable Gauge Theories and Conformal Gravity in Twistor Strings.
(Under the direction of Louise Dolan.)

This dissertation concerns two topics. We first discuss the Yangian structure of deformed integral, four-dimensional $\mathcal{N} = 4$ super Yang-Mills theories using Yangians. We use twist deformations in the Yangian coproducts, which are known to maintain the integrable structure. In a five-state subset of states we examine two explicit cases of deformation resulting in $\mathrm{SU}(2)\times\mathrm{U}(1)^3$ and $\mathrm{SU}(2|1)\times\mathrm{U}(1)^2$, which are subgroups of the $\mathcal{N} = 1$ residual supersymmetry, $\mathrm{PSU}(2,2|1)$, in the full theory. While the full $\mathrm{PSU}(2,2|4)$ Yangian structure is manifest in the deformed theory, we show how the symmetry breaking to $\mathcal{N} = 1$ is produced via twisted coproducts. For the second topic, we display the vertex operators for all states in the conformal supergravity sector of twistor string theory. Using canonical quantization of the open string, we compute N-point tree amplitudes for the supergraviton states. These include amplitudes involving the 'dipole' gravitons, which are not eigenstates of the translation generator. The conformal gravity amplitudes would be hard to access using conventional field theory methods.

Acknowledgments

I would first like to thank my parents, Tim and Diane Ihry. Their love and support has meant more to me than mere words can express. I would like to thank my sister, Liz, for pushing me to strive further and farther. And I would like to thank Eleanor for allowing me to embrace my inner geek.

I would like to thank my advisor Louise Dolan. Her advice, guidance, and support have been invaluable to me. I am also indebted to the numerous professors who have helped shape the physicist I have become.

I am truly grateful to Shaun Gidcumb, Michael Good, Brian Pohl, and Matt Wolboldt for giving me valuable friendship and advice, and for keeping me sane during these turbulent grad school years.

Table of Contents

Abstract **ii**

List of Figures **vi**

List of Tables **vii**

1 Introduction **1**

2 Yangians in Deformed Super Yang-Mills Theories **5**

3 Deformed $\mathcal{N} = 1$ Superconformal Theories **8**

3.1 β Deformed Theory 10

3.2 δ Deformed Theory 12

4 Algebra of the SU$(2|3)$ Subsector **14**

4.1 The SU$(2|3)$ Algebra 15

4.2 The SU$(2|3)$ Yangian Algebra 15

5 Coproducts and Multisite Representations **17**

6 Deforming the Hamiltonian **21**

6.1 The Hamiltonian 21

6.2 The Deformed Hamiltonian 25

6.2.1 Deformed Two-Site Hamiltonian 25

6.2.2 Case 1: β Deformed $\gamma_1 = \gamma_2 = \gamma_3$ 27

6.2.3 Case 2: δ Deformed $\gamma_1 = \gamma_2 = -\gamma_3$ 28

7 Twisted Coproducts **29**

7.1 Case 1: β Deformed $\gamma_1 = \gamma_2 = \gamma_3$ 33

7.2 Case 2: δ Deformed $\gamma_1 = \gamma_2 = -\gamma_3$ 34

8 Conformal Supergravity Tree Amplitudes from Open Twistor Strings **35**

9 Vertex Operators and Canonical Quantization **38**

9.1 Vertex Operators 38

9.2 Norm of the States 45

10 Three Point Couplings **47**

10.1 Unprimed Couplings 48

10.2 Amplitudes with Primed Vertices 61

11 Comparison with Known Conformal and Einstein Amplitudes **68**

12 N-Point Tree Amplitudes for Primed and Unprimed Vertices **74**

13 Conclusion **76**

A Single Index Basis for Yangian Generators **78**

B Double Index Basis for Yangian Generators **81**

C A Subset of Vertex Operators for Calculating Tree Amplitudes **83**

Bibliography **84**

List of Figures

2.1 An open spin chain. 6
6.1 The eigenstate towers of the Hamiltonian. 22

List of Tables

3.1 SU(4) Cartan charges of fields. 10

3.2 U(1) charges of fields under β deformation. 11

3.3 U(1) charges of fields under δ deformation. 13

7.1 Residual and broken symmetries after β deformation. 33

7.2 Residual and broken symmetries after δ deformation. 34

9.1 Vertex operators and helicities for $\mathcal{N} = 4$ conformal supergravity theory 39

9.2 One particle states 45

10.1 Unprimed conformal supergravity MHV couplings 57

10.2 $d = 0$ Unprimed conformal supergravity couplings 60

10.3 MHV Conformal supergravity couplings with primed states 67

10.4 $d = 0$ Conformal supergravity couplings with primed states 67

A.1 Even symmetry generators of SU(2|3). 78

A.2 Odd symmetry generators of SU(2|3) 79

C.1 A subset of the vertex operators: conformal gravitons, scalars and gluons 83

Chapter 1

Introduction

This dissertation will cover two main results: Yangians in Deformed Super Yang-Mills Theories [1] and Conformal Supergravity Tree Amplitudes from Open Twistor String Theory [2].

With the advent of the correspondence between d-dimensional conformal field theory and Anti-deSitter theories, there has been much interest in conformal field theories. An interesting group of superconformal gauge theories has arisen from deformed $\mathcal{N} = 4$ super Yang-Mills theories. In the mid 1990's $\mathcal{N} = 1$ conformal gauge field theories were constructed by exactly marginal deformations [3]. These $\mathcal{N} = 1$ theories have the same particle content as the original $\mathcal{N} = 4$ theory. The $\mathcal{N} = 4$ super Yang-Mills theory was broken to a $\mathcal{N} = 1$ superconformal theory by the addition of a classical marginal deformation with the superpotential. This is often referred to as a beta deformation. Lunin and Maldacena imposed this deformation via a Moyal-like star product [4]. They found the gravity dual of this theory through the AdS/CFT correspondence. A three-parameter family of parameters replacing β was given [5]. Using Bethe ansatz techniques, it was found in [6]-[9] that the one-loop corrections in the large N limit of these deformed theories still provided an integrable spin chain Hamiltonian. These correspond to multi-parameter, $\mathcal{N} = 1$ superconformal theories.

For these theories, amplitudes and finiteness properties have been calculated [10]-[18]. Some further connections between integrability and deformed theories have been discussed in [19]-[23].

In Chapter 2 we review beta and more general twist deformations. In Chapter 3 we discuss $\mathcal{N}=1$ superconformal theories derived from deformed $\mathcal{N}=4$ super Yang-Mills theries. We introduce two explicit examples that will serve as our benchmarks in further chapters. In Chapter 4 we discuss the algebraic structure of SU(2|3) and its Yangian extension. In Chapter 5 we examine coproducts and their use in creating multisite representations of an algebra. In Chapter 6 we give the two-site Hamiltonian as a quadratic Casimir and discuss the SU(2|3) Yangian symmetry of the undeformed theory [26]-[30]. We give the Hamiltonian for the deformed theory in this five-field subsector, in the planar limit, and compute the Yangian generators for various cases including the Lunin-Maldacena deformation. In Chapter 7, we compute the Yangian twisted coproducts associated with multiparameter deformations [31, 32]. This structure was hinted at in [9]. We show that the residual symmetry of the deformed theory continues to use the standard coproduct while the remaining generators do not. We illustrate this in two examples by finding residual $\mathrm{SU}(2)\times\mathrm{U}(1)^3$ and $\mathrm{SU}(2|1)\times\mathrm{U}(1)^2$ symmetry, and discuss how the twisted coproduct is responsible for the smaller symmetry group of the deformed conformal gauge field theory. Finally, we check the conjecture that the Yangian symmetry survives for finite gauge coupling, using the one-loop Hamiltonian and tree level Yangian generators.

In Appendix A we give a single index representation along with the metric and structure constants for such a representation. In Appendix B we show a double index representation and derive the Yangian coproduct structure.

In Chapter 8 we move onto the second topic. We pursue the tree amplitudes for graviton scattering in conformal gravity, described by twistor string theory. The twistor string [33] and its open string formulation [34] describe massless particles of $\mathcal{N} = 4$ Yang-Mills theory coupled to conformal supergravity [35] in four-dimensional Minkowski spacetime.

Conformal gravity field theories [36, 37] provided early examples of finite field theories of gravity [38, 39]. They are not unitary theories, but have interesting structure and continue to provoke comments about possible uses [40]. The equivalence of the twistor string with this field theory system can be exploited to derive conformal gravity tree level scattering amplitudes hard to access in the field theory. In addition, the computation of the gravity trees may be useful in decoupling them from the theory. This would result in a perturbative string theory for super Yang-Mills (with no tower of massive states), and the computational advantage one hopes for in a string theory vs. field theory description. Various efforts towards a QCD string are discussed in [41].

We work in a spinor helicity basis [42]-[44], and compare the conformal gravity tree amplitudes with those of Einstein gravity [45]-[50]. The conformal gravity trees have fewer poles. We compute the conformal couplings in detail, as they should be important in further study of the loop calculation [51].

In Chapter 8 we use the twistor string canonical quantization described in [51, 52] and follow that notation. In Chapter 9, we give the vertex operators for all states in the conformal supergravity multiplets, as suggested by Berkovits and Witten [35]. These include the dipole states, which form pairs of supergravitons in which one state in each pair does not diagonalize the translation generators, and is not a momentum eigenstate.

In Chapter 10, the three-point scattering amplitudes for gluons, gravitons, and scalars, with both one and two negative helicities, are calculated. We include the cases for both members of each dipole, and find a delta function derivative appearing in amplitudes for the members of amplitudes that do not diagonalize the translation generators. These amplitudes still have translation invariance.

In Chapter 11, we extend our results to N-point tree level amplitudes for the diagonal states. We reproduce the Berkovits-Witten formula for maximal helicity violating (MHV) amplitudes for the diagonal states, showing consistency of the canonical approach and the path integral framework. In Chapter 12, we compute the N-point conformal gravity MHV tree amplitudes for gravitons, gluons and scalars in the the full dipole pairs.

Appendix C contains a table of commonly used vertices and should be used as a reference when calculating tree level amplitudes.

Chapter 2

Yangians in Deformed Super Yang-Mills Theories

In 1995 Leigh and Strassler found a method to construct $\mathcal{N} = 1$ superconformal field theories from $\mathcal{N} = 4$ SYM theories via a marginal deformation, so that it remains conformal. This exact marginal deformation, commonly referred to as a beta deformation, takes the form

$$\mathcal{W} = ih\mathrm{Tr}\left[e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2\right] + \frac{ih'}{3}\mathrm{Tr}\left[\Phi_1^3 + \Phi_2^3 + \Phi_3^3\right]. \tag{2.1}$$

The condition for an exact marginal deformation is that the parameters must obey

$$|h|^2\left(1 + \frac{1}{N^2}\left(q - \bar{q}\right)^2\right) + |h'|^2\frac{N^2 - 4}{2N^2} = g^2, \tag{2.2}$$

where h, h', and $q = e^{i\pi\beta}$ are the deformation parameters and g is the Yang-Mills coupling constant. The large N limit for SU(N) gauge theories simplifies this condition, and if we set $h' = 0$ as is commonly done, then $h = g$ is the requirement for a marginal deformation. We will consider real β. Lunin and Maldacena imposed this deformation via a Moyal-like star product [4]

$$X \star Y \equiv e^{i\pi\beta\left(Q^1_X Q^2_Y - Q^2_X Q^1_Y\right)} XY, \tag{2.3}$$

where X, Y are fields in the Lagrangian and the Q's are non R-charge U(1)'s. Frolov found a three parameter family of theories replacing β with β_{ij}. This multiparameter deformation corresponds to the Lagrangian

$$\begin{aligned} \mathcal{L} = \frac{1}{g^2}\mathrm{Tr}\Bigg[&\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \left(D^\mu\bar{\phi}^i\right)\left(D_\mu\phi_i\right) - \frac{1}{2}[\phi_i,\phi_j]_{C_{ij}}[\bar{\phi}^i,\bar{\phi}^j]_{C_{ij}} \\ &+\frac{1}{4}[\phi_i,\bar{\phi}^i][\phi_j,\bar{\phi}^j] + \lambda_A\sigma^\mu D_\mu\bar{\lambda}^A - i([\lambda_4,\lambda_i]_{B_{4i}}\bar{\phi}^i + [\bar{\lambda}^4,\bar{\lambda}^i]_{B_{4i}}\phi_i) \\ &+\frac{i}{2}(\epsilon^{ijk}[\lambda_i,\lambda_j]_{B_{ij}}\Phi_k + \epsilon_{ijk}[\bar{\lambda}^i,\bar{\lambda}^j]_{B_{ij}}\bar{\Phi}^k)\Bigg], \end{aligned} \tag{2.4}$$

where B_{ij} and C_{ij} are related, and describes deformations which generalize γ given in (2.3). The gauge group is SU(N) and the indices run as $1 \leq i, j \leq 3$, since the field content is that of $\mathcal{N} = 4$.

This deformation has been shown to maintain its integrable structure [9]. Therefore in the planar limit we should be able to examine the spin chain of this system. The states of spin chains are single trace product of fields, or letters, on a vacuum state. We can think of the fields as forming a chain of length L, as seen in Figure 2.1. A useful

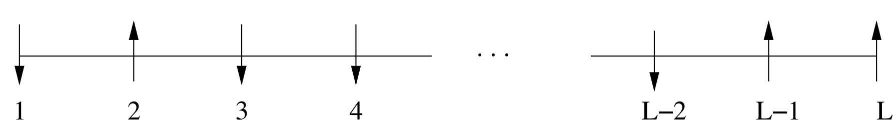

Figure 2.1: An open spin chain.

tool in examining the spin chain structure is the Yangian extension of the symmetry algebra. Since the coproduct of this Hopf algebra can be shown to break the symmetry of the Lagrangian, this process gives a glimpse of exactly how symmetry is broken.

In Chapters 2 - 7 of this dissertation, we calculate the Yangian algebra for this set of deformed Yang-Mills theories. Since these are $\mathcal{N} = 1$ conformal theories, they demonstrate how integrability and the Yangian structure can be useful in more realistic gauge theories.

Chapter 3

Deformed $\mathcal{N} = 1$ Superconformal Theories

The superconformal algebra is $\mathrm{SU}(2,2|1)$, with fifteen even generators of the conformal $\mathrm{SU}(2,2)$ algebra, $(L^{\alpha}{}_{\beta}, L^{\dot{\alpha}}{}_{\dot{\beta}}, K^{\alpha\dot{\beta}}, P_{\alpha\dot{\beta}}, D)$; a $\mathrm{U}(1)_R$ symmetry; and eight supercharges. Leigh and Strassler found a marginal deformotation of the $\mathcal{N} = 4$ superpotential, which results in a $\mathcal{N} = 1$ theory. The superpotential of the original theory

$$\mathcal{W} = ig\mathrm{Tr}\left[\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2\right], \tag{3.1}$$

is replaced by the marginal deformation

$$\mathcal{W} = ig\mathrm{Tr}\left[e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2\right] - ih'\mathrm{Tr}\left[\Phi_1^3 + \Phi_2^3 + \Phi_3^3\right], \tag{3.2}$$

where the Φ_i's are the $\mathcal{N} = 1$ superfields in the adjoint of the gauge group.

Setting $h' = 0$, Lunin and Maldacena found that this type of deformation can be achieved using a Moyal-like star product if we extend the superconformal algebra by two global $\mathrm{U}(1)$'s with charges (Q^1, Q^2). These two charges commute with all the generators of $\mathrm{SU}(2,2|1)$. If we have two fields f and g, their product becomes

$$f \star g = e^{i\pi\beta(Q_f^1 Q_g^2 - Q_f^2 Q_g^1)} fg. \tag{3.3}$$

Beisert and Roiban generated a Moyal-like star product using the charges of the generators of the Cartan subalgebra of the SU(4) symmetry $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$. We label them by (q^1, q^2, q^3), and the product of two fields f and g becomes

$$f \star g = e^{\frac{i}{2}\mathbf{q}_f \times \mathbf{q}_g} fg. \tag{3.4}$$

Using the antisymmetric C-product, one can replace the cross-product of the Cartan charges with

$$\mathbf{q}_f \times \mathbf{q}_g = C_{ab}\, q_f^a\, q_g^b, \tag{3.5}$$

with the C matrix

$$C = \begin{pmatrix} 0 & -\gamma_3 & +\gamma_2 \\ +\gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & +\gamma_1 & 0 \end{pmatrix}. \tag{3.6}$$

At this point, all γ_i are arbitrary, and the deformation is not guaranteed to preserve superconformal symmetry[1]. To achieve a deformed $\mathcal{N} = 1$ theory, we restrict our possible choice of the γ_i so that the product of Cartan charges returns to the product of the two global U(1)'s, as in (3.3).

In this paper we consider the finite subset of states of $\mathrm{SU}(2|3)$, in order to simplify our analysis. The field content is six scalars of the $\mathcal{N} = 4$ multiplet that become three complex bosons of SU(3), $\phi_a = \{\phi_1, \phi_2, \phi_3\}$, and two complex fermions which form a doublet of SU(2), $\psi_\alpha = \{\psi_1, \psi_2\}$. Together they make up the five-state subsector,

[1]But generic γ_i do preserve integrability [9].

$\{\phi_1, \phi_2, \phi_3; \psi_1, \psi_2\}$.

We adopt the oscillator formalism as $\phi_a = c_a^\dagger c_4^\dagger|0\rangle$ and $\psi_\alpha = a_\alpha^\dagger c_4^\dagger|0\rangle$, where the non-zero (anti)commutation relations are $\{c^a, c_b^\dagger\} = \delta_b^a$ and $[a^\alpha, a_\beta^\dagger] = \delta_\beta^\alpha$. We define the three Cartan charges of $\mathrm{SU}(4)_R$ as

$$\mathcal{C}_1 = -R^2{}_2 - R^3{}_3, \quad \mathcal{C}_2 = -R^1{}_1 - R^3{}_3, \quad \mathcal{C}_3 = -R^1{}_1 - R^2{}_2, \tag{3.7}$$

where $R^a{}_b$ are the generators of SU(4) and have the oscillator representation $R^a{}_b = c_b^\dagger c^a - \frac{1}{4}\delta_b^a c_c^\dagger c^c$, where a, b, c run over four indices. We find the values of the Cartan charges on our five fields and list them in Table 3.1.

	q^1	q^2	q^3
ϕ_1	1	0	0
ϕ_2	0	1	0
ϕ_3	0	0	1
ψ_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
ψ_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Table 3.1: SU(4) Cartan charges of fields.

In relation to the fields in the Lagrangian (2.4), $\Phi_A \to \Phi_{ab} \to \Phi_{a4} \equiv \phi_a$ and $\lambda_{4\alpha} = \psi_\alpha$. We now discuss two choices for the set of parameters γ_i that lead to $\mathcal{N} = 1$ superconformal theory, which we will call β and δ deformed theories.

3.1 β Deformed Theory

In the paper by Lunin and Maldacena, the deformation from $\mathcal{N} = 4$ to $\mathcal{N} = 1$ is called the β deformation. The Lagrangian (2.4) preserves two global U(1) transformations, which we list in Table 3.2.

	Q^1	Q^2
$\phi_1,\ \lambda_1$	0	-1
$\phi_2,\ \lambda_2$	1	1
$\phi_3,\ \lambda_3$	-1	0
$A_\mu,\ \lambda_4$	0	0

Table 3.2: U(1) charges of fields under β deformation.

In terms of the Cartan charges, this deformation corresponds to $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$. The C matrix becomes

$$C = \begin{pmatrix} 0 & -\gamma & +\gamma \\ +\gamma & 0 & -\gamma \\ -\gamma & +\gamma & 0 \end{pmatrix}. \tag{3.8}$$

The product of fields f and g becomes

$$\begin{aligned} f \star g &= e^{-\frac{i}{2}\gamma(q_f^1 q_g^2 - q_f^2 q_g^1) - \frac{i}{2}\gamma(q_f^2 q_g^3 - q_f^3 q_g^2) - \frac{i}{2}\gamma(q_f^3 q_g^1 - q_f^1 q_g^3)} \\ &= e^{-\frac{i}{2}\gamma[(q_f^2 - q_f^3)(q_g^2 - q_g^1) - (q_f^2 - q_f^1)(q_g^2 - q_g^3)]}. \end{aligned} \tag{3.9}$$

We identify two U(1)'s based on the Cartan charges, $q^2 - q^3$ and $q^2 - q^1$. Comparing these values with those of Table 3.1 and in the star product (3.3), we find

$$\begin{aligned} U(1)_1 &= \mathcal{C}_2 - \mathcal{C}_3, \\ U(1)_2 &= \mathcal{C}_2 - \mathcal{C}_1, \end{aligned} \tag{3.10}$$

where $\gamma = -2\pi\beta$. We can identify the SU$(2,2|1)$ $\times$U(1)$\times$U(1) generators as the fifteen generators of the conformal group SU$(2,2)$: $L^\alpha{}_\beta, L^{\dot\alpha}{}_{\dot\beta}, K^{\alpha\dot\beta}, P_{\alpha\dot\beta}, D$; a U$(1)_R$ symmetry, $-R^1{}_1 - R^2{}_2 - R^3{}_3$; two global U(1)'s, $U(1)_1 = R^2{}_2 - R^3{}_3$ and $U(1)_2 = R^2{}_2 - R^1{}_1$; and eight supercharges $Q^4{}_\alpha, S^\alpha{}_4, \dot S^{\dot\alpha 4}, \dot Q_{\dot\alpha 4}$.

If we were to denote the star product (3.4) for our five-field subsector,

$$f \star g = e^{iB_{ab}q_f^a q_g^b} fg, \tag{3.11}$$

where $f, g = \{\phi_1, \phi_2, \phi_3; \psi_1\psi_2\}$, then the phase matrix is

$$B = \begin{pmatrix} 0 & -\gamma & +\gamma & 0 & 0 \\ +\gamma & 0 & -\gamma & 0 & 0 \\ -\gamma & +\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.12}$$

As we shall see later, in this subsector this deformation will leave residual $\mathrm{SU}(2)\times\mathrm{U}(1)^3$ symmetry, a subgroup of $\mathrm{SU}(2,2|1) \times \mathrm{U}(1) \times \mathrm{U}(1)$.

3.2 δ Deformed Theory

We now show a different superconformal $\mathcal{N} = 1$ deformed theory, in which we first choose our phase matrix and then work in reverse to find the corresponding global U(1) charges. Let us choose the phase elements $\gamma_1 = \gamma_2 = -\gamma_3 = \gamma$. The phase matrix is

$$C = \begin{pmatrix} 0 & +\gamma & +\gamma \\ -\gamma & 0 & -\gamma \\ -\gamma & +\gamma & 0 \end{pmatrix}. \tag{3.13}$$

The product of fields f and g is

$$\begin{aligned} f \star g &= e^{-\frac{i}{2}\gamma(q_f^2 q_g^1 - q_f^1 q_g^2) - \frac{i}{2}\gamma(q_f^3 q_g^1 - q_f^1 q_g^3) - \frac{i}{2}\gamma(q_f^2 q_g^3 - q_f^3 q_g^2)} fg \\ &= e^{\frac{i}{2}\gamma((q_f^2 - q_f^1)(q_g^1 + q_g^3) - (q_f^1 + q_f^3)(q_g^2 - q_g^1))} fg \end{aligned} \tag{3.14}$$

The two U(1)'s are $U(1)_1 = \mathcal{C}_2 - \mathcal{C}_1$ and $U(1)_2 = \mathcal{C}_1 + \mathcal{C}_3$ and $\gamma = -2\pi\beta$. Using Table 3.1, we find the values of the U(1) charges listed in Table 3.3. The U(1)'s are

	Q^1	Q^2
ϕ_1, λ_1	-1	1
ϕ_2, λ_2	1	0
ϕ_3, λ_3	0	1
A_μ, λ_4	0	1

Table 3.3: U(1) charges of fields under δ deformation.

different from the prior case, leading to a different $\mathcal{N}=1$ theory with two global U(1)'s. Now we have the fifteen generators of the conformal algebra, $L^\alpha{}_\beta, L^{\dot\alpha}{}_{\dot\beta}, K^{\alpha\dot\beta}, P_{\alpha\dot\beta}$, and D; a $\mathrm{U}(1)_R$ symmetry, $-R^1{}_1 - R^2{}_2 - R^4{}_4$; two global U(1)'s $\mathrm{U}(1)_1 = R^2{}_2 - R^1{}_1$ and $\mathrm{U}(1)_2 = R^4{}_4 - R^2{}_2$; and eight supercharges $Q^3{}_\alpha, S^\alpha{}_3, \dot{Q}_{\dot\alpha 3}, \dot{S}^{\dot\alpha 3}$, for a total of eighteen even and eight odd generators.

If we examine the five field content of the SU(2|3) subsector, the product of two fields, f and g, appear as (3.11); however, the phase matrix has changed to

$$B = \begin{pmatrix} 0 & +\gamma & +\gamma & +\gamma & +\gamma \\ -\gamma & 0 & -\gamma & -\gamma & -\gamma \\ -\gamma & +\gamma & 0 & 0 & 0 \\ -\gamma & +\gamma & 0 & 0 & 0 \\ -\gamma & +\gamma & 0 & 0 & 0 \end{pmatrix}. \tag{3.15}$$

As we shall see later, this results in a residual SU(2|1) $\times$ U(1) $\times$ U(1) symmetry to the SU(2|3) subsector. This has a richer structure than the β-deformed case, after the restriction to our subsector, since we are left with an unbroken *super*algebra.

Chapter 4

Algebra of the SU$(2|3)$ Subsector

We review the fields of this closed subsector and discuss its symmetry algebra. The five fields include two complex fermions and three complex bosons; $\mathbf{\Phi}_I = \{\psi_1, \psi_2; \phi_1, \phi_2, \phi_3\}$. We can express these fields as single particle states,

$$\phi_a(i)|0\rangle = c_a^\dagger(i) c_4^\dagger(i)|0\rangle, \quad \psi_\alpha(j)|0\rangle = a_\alpha^\dagger(j) c_4^\dagger(j)|0\rangle, \tag{4.1}$$

where $1 \leq \alpha, \beta \leq 2$ and $1 \leq a, b \leq 3$ unless otherwise stated. Site indices i, j run over the length of the chain; $1 \leq i, j \leq L$. The oscillator (field), $c_4^\dagger(i)$, is a remnant of the full PSU$(2,2|4)$ theory [24] and is included to ensure the fermionic and bosonic properties of this oscillator representation. The oscillator commutation relations are

$$[a^\alpha(i), a_\beta^\dagger(j)] = \delta_\beta^\alpha \delta_j^i, \quad \{c^a(i), c_b^\dagger(j)\} = \delta_b^a \delta_j^i. \tag{4.2}$$

The twenty-four generators of the SU$(2|3)$ superalgebra have the explicit representation, at tree level $(g = 0)$,

$$\begin{aligned} R^a{}_b &= c_b^\dagger c^a - \frac{1}{3}\delta_b^a c_c^\dagger c^c, \quad L^\alpha{}_\beta = a_\beta^\dagger a^\alpha - \frac{1}{2}\delta_\beta^\alpha a_\gamma^\dagger a^\gamma, \\ D &= c_c^\dagger c^c + \frac{3}{2} a_\gamma^\dagger a^\gamma, \quad S^\gamma{}_c = c_c^\dagger a^\gamma, \quad Q^c{}_\gamma = a_\gamma^\dagger c^c. \end{aligned} \tag{4.3}$$

4.1 The SU$(2|3)$ Algebra

A single index basis for the symmetry generators of the ordinary SU(2|3) algebra is given in Appendix A. The symmetry generators for SU(3) and SU(2) carry indices $\{1,\ldots,8\}$ and $\{9,10,11\}$, respectively; the dilation generator has index 12, and the odd generators are labeled by $\{13,\ldots,24\}$. A detailed analysis of the SU(2|3) algebra and resulting spin chain can be found in [24, 25]. The symmetry generators close the algebra

$$\left[J^A, J^B\right\} = f^{AB}{}_C J^C = f^{ABD} g_{DC} J^C. \tag{4.4}$$

An explicit list of the structure constants f^{ABC} and the metric g_{AB} of SU(2|3) can be found in Appendix A. This basis allows for a simple presentation of the Yangian defining relations.

4.2 The SU$(2|3)$ Yangian Algebra

An infinite-dimensional extension of the SU(2|3) algebra, called the Yangian [54, 53], has a tree level representation in terms of the ordinary generators

$$Q_0^A = -f^A{}_{CB} \sum_{i<j} J_0^B(i) J_0^C(j). \tag{4.5}$$

This representation takes into account the superalgebra properties of the Lie algebra [26]. The super Yangian algebra defining relations are

$$\left[J^A, J^B\right\} = f^{AB}{}_C J^C, \tag{4.6}$$

$$\left[J^A, Q^B\right\} = f^{AB}{}_C Q^C, \tag{4.7}$$

$$\left[Q^{[A}, \left[Q^B, J^{C]}\right\}\right\} = \alpha f^{AG}{}_D f^{BH}{}_E f^{CK}{}_F f_{GHK} J^{\{D} J^E J^{F\}}. \tag{4.8}$$

The last is the Serre relation, which holds because the generators J^A are in a certain representation. The constant α depends on the normalization of the basis. Here $J^{\{D}J^E J^{F\}}$ is the totally symmetric product, with an additional minus sign for the exchange of two odd generators.

Chapter 5

Coproducts and Multisite Representations

A coproduct is a holomorphic map $\Delta : \mathcal{A} \to \mathcal{A}$. This construct establishes a method of building a two-site representation from a single-site, a three-site from a two-site, etc. In this paper we use the Yangian algebra of SU(2|3) written as $\mathcal{Y}($ SU(2|3) $)$. The definition of the coproduct for the algebraic element J^A is

$$\Delta J^A = J^A \otimes \mathcal{I} + \mathcal{I} \otimes J^A. \tag{5.1}$$

In addition, $\Delta\Delta = \mathcal{I}$ and $\Delta\mathcal{I} = \mathcal{I} \otimes \mathcal{I}$. As this is a coassociative, algebraic structure, we are given some leeway in the defining of the above. In this chapter we shall use a subscript to denote the 'dimension' or number of sites in a given generator. E.g. $\mathcal{I} \otimes \mathcal{I} \otimes \mathcal{I} = \mathcal{I}_3$, where $\mathcal{I}$ is the identity.

To get an $(n+1)$-site ordinary generator,

$$J^A_{n+1} = \Delta J^A_n = J^A_1 \otimes \mathcal{I}_n + \mathcal{I}_1 \otimes J^A_n. \tag{5.2}$$

Creating a two-site representation from a one-site,

$$
\begin{aligned}
J_2^A = \Delta J_1^A &= J_1^A \otimes \mathcal{I}_1 + \mathcal{I}_1 \otimes J_1^A \\
&= J^A(1) + J^A(2).
\end{aligned}
\tag{5.3}
$$

Starting from a single-site representation, we have all one-dimensional elements in our product. The second line contains the more commonly used $J^A(i)$ representation, where i is the site index. While this is a more comfortable notation, while examining twists we must use the product structure to get the twists, then we can use resume with the site index notation.

Building to three-sites, we tensor one-dimensional pieces to two-dimensional pieces

$$
\begin{aligned}
J_3^A = \Delta J_2^A &= J_1^A \otimes \mathcal{I}_2 + \mathcal{I}_1 \otimes J_2^A \\
&= J^A \otimes \mathcal{I} \otimes \mathcal{I} + \mathcal{I} \otimes \left(J^A \otimes \mathcal{I} + \mathcal{I} \otimes J^A\right) \\
&= J^A \otimes \mathcal{I} \otimes \mathcal{I} + \mathcal{I} \otimes J^A \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{I} \otimes J^A \\
&= J^A(1) + J^A(2) + J^A(3).
\end{aligned}
\tag{5.4}
$$

The L-site representation is built much the same as the above:

$$
\begin{aligned}
J_{L+1}^A = \Delta J_L^A &= J_1^A \otimes \mathcal{I}_L + \mathcal{I}_1 \otimes J_L^A \\
&= J^A \otimes (\mathcal{I} \otimes \cdots \otimes \mathcal{I}) + \mathcal{I} \otimes \left(J^A \otimes \mathcal{I}_{L-1} + \cdots + \mathcal{I} \otimes J_{L-1}^A\right) \\
&= J^A \otimes \cdots \otimes \mathcal{I} + \mathcal{I} \otimes J^A \otimes \cdots \otimes \mathcal{I} + \cdots + \mathcal{I} \otimes \cdots \otimes J^A \\
&= J^A(1) + J^A(2) + \cdots + J^A(L+1).
\end{aligned}
\tag{5.5}
$$

Yangian coproducts require three pieces:

$$
\Delta Q^A = Q^A \otimes \mathcal{I} + \mathcal{I} \otimes Q^A - f^A{}_{CB} J^B \otimes J^C.
\tag{5.6}
$$

We recognize the third piece as that given in the previous chapter for the two-site Yangian generator $Q_2^A = -f^A{}_{CB}J^BJ^A$. In agreement with the ordinary generators, we tensor one-dimensional to n-dimensional representations to create an (n+1)-dimensional Yangian representation

$$Q_{n+1}^A = \Delta Q_n^A = Q_1^A \otimes \mathcal{I}_n + \mathcal{I}_1 \otimes Q_n^A - f^A{}_{CB}J_1^B \otimes J_n^C. \tag{5.7}$$

There is no one-site Yangian, $Q_1^A = 0$. The two-site Yangian is a simple product of the final term

$$\begin{aligned}
Q_2^A &= \Delta Q_1^A \\
&= Q_1^A \otimes \mathcal{I}_1 + \mathcal{I}_1 \otimes Q_1^A - f^A{}_{CB}J_1^B \otimes J_1^C \\
&= -f^A{}_{CB}J^B \otimes J^C \\
&= -f^A{}_{CB}J^B(1)J^C(2) \\
&= Q^A(1,2).
\end{aligned} \tag{5.8}$$

Creating a three-site Yangian,

$$\begin{aligned}
Q_3^A &= \Delta Q_2^A \\
&= \mathcal{I} \otimes Q_2^A - f^A{}_{CB}J_1^B \otimes J_2^C \\
&= \mathcal{I} \otimes \left(-f^A{}_{CB}J^B \otimes J^C\right) - f^A{}_{CB}J^B \otimes \left(J^C \otimes \mathcal{I} + \mathcal{I} \otimes J^C\right) \\
&= -f^A{}_{CB}\left(\mathcal{I} \otimes J^B \otimes J^C + J^B \otimes J^C \otimes \mathcal{I} + J^B \otimes \mathcal{I} \otimes J^C\right) \\
&= Q^A(2,3) + Q^A(1,2) + Q^A(1,3) \\
&= \sum_{1 \le i < j \le 3} Q^A(i,j).
\end{aligned} \tag{5.9}$$

This fits the structure $Q^A = \sum_{i<j} Q^A(i,j)$. And we can create an L-site Yangian

representation as

$$
\begin{aligned}
Q^A_{L+1} &= \Delta Q^A_L \\
&= \mathcal{I} \otimes Q^A_L - f^A{}_{CB} J^B \otimes J^C_L \\
&= \sum_{1<i<j} Q^A(i,j) - f^A{}_{CB} J^B \otimes \left(J^C \otimes \cdots \otimes \mathcal{I} + \cdots + \mathcal{I} \otimes \cdots \otimes J^C \right) \\
&= \sum_{1<i<j} Q^A(i,j) + Q^A(1,2) + Q^A(1,3) + \cdots + Q^A(1,L+1) \\
&= \sum_{i<j} Q^A(i,j). \qquad (5.10)
\end{aligned}
$$

We have used a single index basis for the generators; however, we could have used the equivalent double index counterparts. Later it shall be necessary to to adopt the double indices, but the mechanics of the construction remain the same.

Chapter 6

Deforming the Hamiltonian

6.1 The Hamiltonian

A useful feature of this sector is the relationship between the quadratic Casimir and the two-site Hamiltonian. A Hamiltonian, of generic length L, was found in [24]. The two-site Hamiltonian is

$$\begin{aligned} H(1,2) = & \left(c_a^\dagger(1)c_b^\dagger(2) - c_b^\dagger(1)c_a^\dagger(2)\right) c^b(2)c^a(1) \\ & + \left(c_a^\dagger(1)a_\alpha^\dagger(2) + a_\alpha^\dagger(1)c_a^\dagger(2)\right) a^\alpha(2)c^a(1) \\ & + \left(a_\alpha^\dagger(1)c_a^\dagger(2) + c_a^\dagger(1)a_\alpha^\dagger(2)\right) c^a(2)a^\alpha(1) \\ & + \left(a_\alpha^\dagger(1)a_\beta^\dagger(2) + a_\beta^\dagger(1)a_\alpha^\dagger(2)\right) a^\beta(2)a^\alpha(1). \end{aligned} \tag{6.1}$$

One can explicitly check that the Hamiltonian above has two eigenstates with eigenvalues 0 and 2. These correspond to symmetric and antisymmetric two-particle states and are discussed below.

The two-site quadratic Casimir is the operator $g_{AB}J^AJ^B = g_{AB}(J(1)^A + J^A(2))(J^B(1) + J^B(2))$. And it can be explicitly shown that,

$$g_{AB}J^AJ^B = \tfrac{1}{3}D^2 + \tfrac{1}{2}L^\gamma{}_\delta L^\delta{}_\gamma - \tfrac{1}{2}R^c{}_d R^d{}_c - \tfrac{1}{2}\left[Q^c{}_\gamma, S^\gamma{}_c\right]. \tag{6.2}$$

The single-site quadratic Casimir acting on any two-particle state $|\eta\rangle$ is zero. So, $g_{AB}J^A(1)J^B(1)|\eta\rangle = g_{AB}J^A(2)J^B(2)|\eta\rangle = 0$, and the cross term piece is $2g_{AB}J^A(1)J^B(2) = H(1,2)$, where $H(1,2)$ is given in (6.1). The two-site Hamiltonian can be identified with the quadratic Casimir

$$H(1,2)|\eta\rangle = g_{AB}[J^A(1) + J^A(2)][J^B(1) + J^B(2)]|\eta\rangle, \tag{6.3}$$

when acting on the states. For calculations with Yangians it is useful to use eigenstates

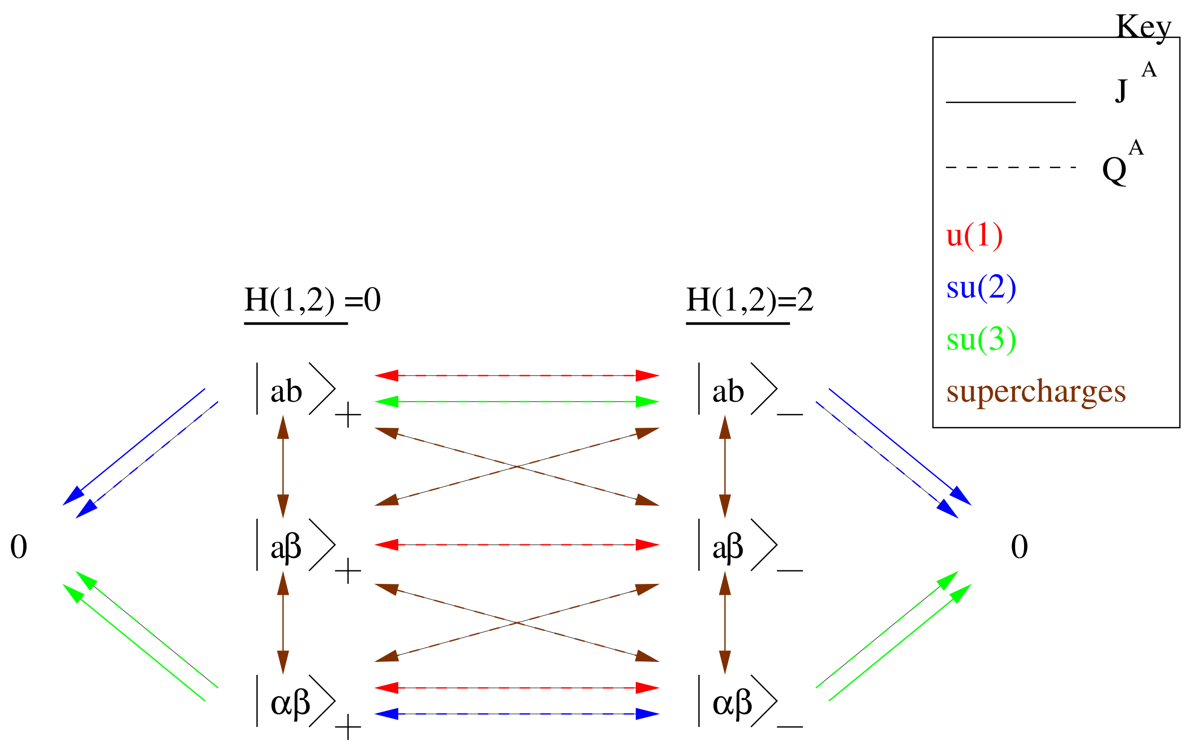

Figure 6.1: The eigenstate towers of the Hamiltonian.

of the Hamiltonian. The two-particle eigenstates are symmetric or antisymmetric in the site indices. We define them as $|\mathbf{\Phi}_I\mathbf{\Phi}_J\rangle_\pm = |\mathbf{\Phi}_I(1)\mathbf{\Phi}_J(2)\rangle \pm |\mathbf{\Phi}_I(2)\mathbf{\Phi}_J(1)\rangle$. The explicit representation of the symmetric states is

$$
\begin{aligned}
|ab\rangle_+ &= -\left(c_a^\dagger(1)c_b^\dagger(2) + c_b^\dagger(1)c_a^\dagger(2)\right) c_4^\dagger(1)c_4^\dagger(2)|0\rangle, \\
|a\beta\rangle_+ &= \left(c_a^\dagger(1)a_\beta^\dagger(2) - a_\beta^\dagger(1)c_a^\dagger(2)\right) c_4^\dagger(1)c_4^\dagger(2)|0\rangle, \\
|\alpha\beta\rangle_+ &= \left(a_\alpha^\dagger(1)a_\beta^\dagger(2) - a_\beta^\dagger(1)a_\alpha^\dagger(2)\right) c_4^\dagger(1)c_4^\dagger(2)|0\rangle.
\end{aligned} \tag{6.4}
$$

The representation of the antisymmetric states is

$$
\begin{aligned}
|ab\rangle_- &= -\left(c_a^\dagger(1)c_b^\dagger(2) - c_b^\dagger(1)c_a^\dagger(2)\right) c_4^\dagger(1)c_4^\dagger(2)|0\rangle, \\
|a\beta\rangle_- &= \left(c_a^\dagger(1)a_\beta^\dagger(2) + a_\beta^\dagger(1)c_a^\dagger(2)\right) c_4^\dagger(1)c_4^\dagger(2)|0\rangle, \\
|\alpha\beta\rangle_- &= \left(a_\alpha^\dagger(1)a_\beta^\dagger(2) + a_\beta^\dagger(1)a_\alpha^\dagger(2)\right) c_4^\dagger(1)c_4^\dagger(2)|0\rangle,
\end{aligned} \tag{6.5}
$$

These two groups, symmetric and antisymmetric, make up two towers. Ordinary symmetry generators of SU(2|3), move one up and down in each tower, while Yangian generators move from one tower to a linear combination in the other.

Commutation of the dilatation operator with the other symmetry generators gives the anomolous dimension of that operator: $[D, J^A] = (\mathrm{dim}J^A)J^A$. This relation holds for the Yangian: $[D, Q^A] = (\mathrm{dim}J^A)Q^A$. Assuming these relations hold to all orders of the Yang-Mills coupling, as discussed in [28], we expand the operators,

$$
\begin{aligned}
\left[D, Q^A\right] &= \left[D_0 + g_{\mathrm{YM}}^2 D_2 + \mathcal{O}(g_{YM}^3), Q_0^A + g_{\mathrm{YM}}Q_1^A + g_{\mathrm{YM}}^2 Q_2^A + \mathcal{O}(g_{YM}^3)\right] \\
&= \left[D_0, Q_0^A + g_{\mathrm{YM}}Q_1^A + g_{\mathrm{YM}}^2 Q_2^A\right] + g_{YM}^2\left[D_2, Q_0^A\right] + \mathcal{O}(g_{YM}^3) \\
&= (\mathrm{dim}J^A)(Q_0^A + g_{\mathrm{YM}}Q_1^A + g_{\mathrm{YM}}^2 Q_2^A) + g_{\mathrm{YM}}^2[D_2, Q_0^A] + \mathcal{O}(g_{YM}^3).
\end{aligned} \tag{6.6}
$$

We find that $[D_2, Q_0^A]$ must be zero. In PSU(2, 2|4), an explicit check of the commutator gives the lattice derivative or ‘edge effects’ of the system, $[D_2, Q_0^A] = q^A \sim 0$, where

$q^A(1,L) = J^A(1) - J^A(L)$. Our Yangian in SU(2|3), while not a subalgebra of the Yangian of PSU(2,2|4), maintains this relation. To see this we introduce the identity

$$\left[g_{AB}J(1)^A J(2)^B, q^C(1,2)\right] = 4Q^C(1,2), \tag{6.7}$$

where $Q^A(1,2)$ is the two-site version of the bare generator Q_0^A. So we have $\frac{1}{4}\left[H(1,2), q^A(1,2)\right] = Q^A(1,2)$, and the one-loop calculation becomes

$$\begin{aligned}\left[H(1,2), Q^A(1,2)\right] = &\\ &\tfrac{1}{4}\left(H(1,2)^2 q^A(1,2) - 2H(1,2)q^A(1,2)H(1,2) + q^A(1,2)H(1,2)^2\right).\end{aligned} \tag{6.8}$$

From Fig. 6.1, we see that the Yangian acting on a tower moves it to the other tower (i.e. moves a symmetric state to an antisymmetric and vice versa). From (6.7), if the Yangian produces this type of movement so must the edge effect, q^A. Recalling the values under the Hamiltonian of the two-particle states $H(1,2)|\mathbf{\Phi}_1\mathbf{\Phi}_2\rangle_+ = 0$, $H(1,2)|\mathbf{\Phi}_1\mathbf{\Phi}_2\rangle_- = 2|\mathbf{\Phi}_1\mathbf{\Phi}_2\rangle_-$, we see that the middle term vanishes under both symmetric and antisymmetric states. We find that

$$\left[H(1,2), Q^A(1,2)\right]|\mathbf{\Phi}_1\mathbf{\Phi}_2\rangle_\pm = q^A(1,2)|\mathbf{\Phi}_1\mathbf{\Phi}_2\rangle_\pm, \tag{6.9}$$

which is the two-site version of the edge effect described above in the SU(2|3) sector.

6.2 The Deformed Hamiltonian

We turn to the twist deformation found in [9]. A second solution to the graded Yang-Baxter equation for the SU(2|3) sector was given:

$$\tilde{R} = \frac{1}{u+i}\left(ue^{-iB_{ij}}\mathcal{I}_{ij}^{kl} + i\mathcal{P}_{ij}^{kl}\right). \tag{6.10}$$

This deformed R-matrix is the conventional R-matrix solution to the Yang-Baxter equation with additional phases. The identity and projection operators are $\mathcal{I}_{ij}^{kl} = \delta_i^k\delta_j^l$ and $\mathcal{P}_{ij}^{kl} = \delta_i^l\delta_j^k$. The deformed monodromy matrix is defined

$$\tilde{\mathcal{T}}_{a;\alpha_1\ldots\alpha_L}^{b;\beta_1\ldots\beta_L} = \tilde{R}_{a\alpha_L}^{b_{L-1}\beta_L}\tilde{R}_{b_{L-1}\alpha_{L-1}}^{b_{L-2}\beta_{L-1}}\cdots\tilde{R}_{b_2\alpha_2}^{b_1\beta_2}\tilde{R}_{b_1\alpha_1}^{a\beta_1}\exp\left[i\pi\sum_{i=1}^{L}\sum_{j=1}^{i-1}([\alpha_i]+[\beta_i])[\alpha_j]\right], \tag{6.11}$$

where the $\mathbb{Z}_2$ graded set of states is denoted by the $[\alpha_i]$. Derived from the trace of the monodromy matrix, the deformed transfer matrix is $\tilde{\mathcal{T}}(u) = (-)^{[a]}\tilde{\mathcal{T}}_a^a(u)$. As in the normal case, we find that the deformed Hamiltonian is the logarithmic derivative of the deformed transfer matrix, $\tilde{\mathcal{H}} = -i\left(\tilde{\mathcal{T}}(u^*)\right)^{-1}\frac{d}{du}\tilde{\mathcal{T}}(u)\Big|_{u=u^*}$. This is more closely examined for two-particle states in the following section.

6.2.1 Deformed Two-Site Hamiltonian

The two-site transfer matrix is $\tilde{T}(u) = \tilde{R}_{a\alpha_2}^{b_1\beta_2}\tilde{R}_{b_1\alpha_1}^{a\beta_1}\exp\left[i\pi([\alpha_2]+[\beta_2])[\alpha_1]\right]$. Calculating the Hamiltonian, we find the logarithmic derivative of the deformed transfer matrix and expand at $u^* = 0$,

$$\begin{aligned}\tilde{\mathcal{H}} &= \left(\delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2} - \delta_{\alpha_1}^{\beta_2}\delta_{\alpha_2}^{\beta_1}e^{-iB_{\alpha_1\alpha_2}}\right) + \left(\delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2} - \delta_{\alpha_1}^{\beta_2}\delta_{\alpha_2}^{\beta_1}e^{-iB_{\alpha_2\alpha_1}}\right)\\ &= \left(\tilde{\mathcal{H}}_{\alpha_1\alpha_2}^{\beta_1\beta_2}\right) + \left(\tilde{\mathcal{H}}_{\alpha_2\alpha_1}^{\beta_2\beta_1}\right).\end{aligned} \tag{6.12}$$

We define the deformed two-site Hamiltonians, $\tilde{H}(1,2) \equiv \tilde{\mathcal{H}}^{\beta_1\beta_2}_{\alpha_1\alpha_2}$ and $\tilde{H}(2,1) \equiv \tilde{\mathcal{H}}^{\beta_2\beta_1}_{\alpha_2\alpha_1}$. Examining more closely, we see that a phase is obtained under interchange of two fields.

$$\begin{aligned}\tilde{H}(1,2) = &\left(c^\dagger_a(1)c^\dagger_b(2) - e^{-iB_{ab}}c^\dagger_b(1)c^\dagger_a(2)\right) c^b(2)c^a(1) \\ &+ \left(c^\dagger_a(1)a^\dagger_\alpha(2) + e^{-iB_{a\alpha}}a^\dagger_\alpha(1)c^\dagger_a(2)\right) a^\alpha(2)c^a(1) \\ &+ \left(a^\dagger_\alpha(1)c^\dagger_a(2) + e^{-iB_{\alpha a}}c^\dagger_a(1)a^\dagger_\alpha(2)\right) c^a(2)a^\alpha(1) \\ &+ \left(a^\dagger_\alpha(1)a^\dagger_\beta(2) + e^{-iB_{\alpha\beta}}a^\dagger_\beta(1)a^\dagger_\alpha(2)\right) a^\beta(2)a^\alpha(1).\end{aligned} \tag{6.13}$$

The antisymmetric matrix B_{AB} is a matrix of phases formed from the charges of the Cartan generators of the original R symmetry SU(4) and is described in [9],

$$\mathbf{B} = \begin{pmatrix} 0 & -\gamma_3 & +\gamma_2 & \frac{1}{2}(\gamma_2-\gamma_3) & \frac{1}{2}(\gamma_2-\gamma_3) \\ +\gamma_3 & 0 & -\gamma_1 & \frac{1}{2}(\gamma_3-\gamma_1) & \frac{1}{2}(\gamma_3-\gamma_1) \\ -\gamma_2 & +\gamma_1 & 0 & \frac{1}{2}(\gamma_1-\gamma_2) & \frac{1}{2}(\gamma_1-\gamma_2) \\ \frac{1}{2}(\gamma_3-\gamma_2) & \frac{1}{2}(\gamma_1-\gamma_3) & \frac{1}{2}(\gamma_2-\gamma_1) & 0 & 0 \\ \frac{1}{2}(\gamma_3-\gamma_2) & \frac{1}{2}(\gamma_1-\gamma_3) & \frac{1}{2}(\gamma_2-\gamma_1) & 0 & 0 \end{pmatrix}. \tag{6.14}$$

The deformation parameters γ_i are three real constants. The eigenstates of the deformed Hamiltonian are

$$\begin{aligned}\widetilde{|ab\rangle}_\pm &= -\left(e^{iB_{ab}/2}c^\dagger_a(1)c^\dagger_b(2) \pm e^{-iB_{ab}/2}c^\dagger_b(1)c^\dagger_a(2)\right) c^\dagger_4(1)c^\dagger_4(2)|0\rangle, \\ \widetilde{|a\beta\rangle}_\pm &= \left(e^{iB_{a\beta}/2}c^\dagger_a(1)a^\dagger_\beta(2) \mp e^{-iB_{a\beta}/2}a^\dagger_\beta(1)c^\dagger_a(2)\right) c^\dagger_4(1)c^\dagger_4(2)|0\rangle, \\ \widetilde{|\alpha\beta\rangle}_\pm &= \left(e^{iB_{\alpha\beta}/2}a^\dagger_\alpha(1)a^\dagger_\beta(2) \mp e^{-iB_{\alpha\beta}/2}a^\dagger_\beta(1)a^\dagger_\alpha(2)\right) c^\dagger_4(1)c^\dagger_4(2)|0\rangle.\end{aligned} \tag{6.15}$$

As before, they have eigenvalues $\tilde{H}_2\widetilde{|+\rangle} = 0\widetilde{|+\rangle}$ and $\tilde{H}_2\widetilde{|-\rangle} = 2\widetilde{|-\rangle}$. Note that special cases of repeated fields will never receive phase corrections.

6.2.2 Case 1: β Deformed $\gamma_1 = \gamma_2 = \gamma_3$

Since we are interested in deformed theories that have $\mathcal{N} = 1$ superconformal symmetry, we first examine the case of phase deformations in which all parameters are equal, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$. The resultant nonzero phases are $B_{13} = B_{21} = B_{32} = \gamma$ and give a residual SU(2)$\times$U(1)^3 symmetry. This is the beta deformation of Lunin and Maldacena [4, 9], but restricted to our five field subsector. So the nonzero commutation relations after deformation are just the SU(2) algebra

$$[L^\alpha{}_\beta, L^\gamma{}_\delta] = \delta^\alpha_\delta L^\gamma{}_\beta - \delta^\gamma_\beta L^\alpha{}_\delta. \tag{6.16}$$

The three U(1) generators are $U_1 = \frac{3}{4}c_4^\dagger c^4 - \frac{1}{4}c_c^\dagger c^c$, $U_2 = c_2^\dagger c^2 - c_3^\dagger c^3$, and $U_3 = c_2^\dagger c^2 - c_1^\dagger c^1$. Two-particle eigenstates of the deformed Hamiltonian have a phase on the states with two SU(3) fields and no phase on any of the additional states.

$$\begin{aligned}
\widetilde{|ab\rangle}_\pm &= -\left(e^{iB_{ab}/2}c_a^\dagger(1)c_b^\dagger(2) \pm e^{-iB_{ab}/2}c_b^\dagger(1)c_a^\dagger(2)\right)c_4^\dagger(1)c_4^\dagger(2)|0\rangle,\\
\widetilde{|a\beta\rangle}_\pm &= \left(c_a^\dagger(1)a_\beta^\dagger(2) \mp a_\beta^\dagger(1)c_a^\dagger(2)\right)c_4^\dagger(1)c_4^\dagger(2)|0\rangle,\\
\widetilde{|\alpha\beta\rangle}_\pm &= \left(a_\alpha^\dagger(1)a_\beta^\dagger(2) \mp a_\beta^\dagger(1)a_\alpha^\dagger(2)\right)c_4^\dagger(1)c_4^\dagger(2)|0\rangle.
\end{aligned} \tag{6.17}$$

If we tried to examine the one-loop quantity $[\tilde{H}, Q_0^A]$ using the Yangian from the undeformed SU(2|3) theory, we would find

$$\begin{gathered}
\left[\tilde{H}, Q^A\right]\widetilde{|ab\rangle}_\pm = q^A\widetilde{|ab\rangle}_\pm, \quad \left[\tilde{H}, Q^A\right]\widetilde{|a\alpha\rangle}_\pm = q^A\widetilde{|a\alpha\rangle}_\pm,\\
\left[\tilde{H}, Q^A\right]\widetilde{|\alpha\beta\rangle}_\pm = q^A\widetilde{|\alpha\beta\rangle}_\pm,
\end{gathered} \tag{6.18}$$

only for $A = \{A|J^A \in SU(2) \times \mathrm{U}(1)^3\}$.

6.2.3 Case 2: δ Deformed $\gamma_1 = \gamma_2 = -\gamma_3$

Another $\mathcal{N} = 1$ superconformal theory, embedded differently in the original PSU$(2, 2|4)$ algebra, is given by $\gamma_1 = \gamma_2 = -\gamma_3$. The nonzero elements of the antisymmetric matrix are $B_{ab} : B_{12} = B_{13} = -B_{23} = \gamma$ and $B_{a\alpha} : B_{1\alpha} = -B_{2\beta} = \gamma$.[1] The residual symmetry is SU$(2|1)$ $\times$U$(1)^2$. This symmetry algebra has a richer structure, containing a superalgebra containing $\{L^\alpha{}_\beta, Q^3{}_\alpha, S^\alpha{}_3, R\}$ and the two U(1)'s: $R = a^\dagger_\gamma a^\gamma + 2c^\dagger_c c^c$, $U_2 = c^\dagger_2 c^2 - c^\dagger_1 c^1$, $U_3 = c^\dagger_4 c^4 - c^\dagger_2 c^2$. The nonzero commutation relations for this form of the embedding are

$$
\begin{gathered}
[L^\alpha{}_\beta, J_\gamma] = \delta^\alpha_\gamma J_\beta - \frac{1}{2}\delta^\alpha_\beta J_\gamma, \quad [L^\alpha{}_\beta, J^\gamma] = -\delta^\gamma_\beta J_\alpha + \frac{1}{2}\delta^\alpha_\beta J^\gamma, \\
[R, S^\alpha{}_3] = S^\alpha{}_3, \quad \left[R, Q^3{}_\alpha\right] = -Q^3{}_\alpha, \quad \left\{S^\alpha{}_3, Q^3{}_\beta\right\} = L^\alpha{}_\beta + \frac{1}{2}\delta^\alpha_\beta R.
\end{gathered} \tag{6.19}
$$

We could again try to compute with the tree level Yangian in the deformed theory; however we would find that unless we use Q^A with $\{A \in \text{SU}(2|1) \times \text{U}(1)^2\}$ and restrict to eigenstates whose one particle fields lie in the fundamental representation of the residual symmetry, the standard form of the tree level Yangian (4.5) is not useful.

Therefore, we look for the appropriate form of the tree level Yangian from the deformed transfer matrix. In [32], a twisted R-matrix is derived via a Reshetikhin twist [31], leading to a deformed coproduct. Our deformed R-matrix is a supersymmetric version of this, as briefly mentioned in [9]. So we will use a twisted coproduct to compute the tree level Yangian.

[1]For the remainder of this section, $1 \le a, b \le 2$, $1 \le \alpha, \beta \le 2$.

Chapter 7

Twisted Coproducts

We identify the deformed R-matrix in Eq. (6.10) with a multiparameter form [31, 32]. This requires a twisted coproduct on our generators. For an algebra $\mathcal{A}$, a coproduct is a homomorphic map $\Delta : \mathcal{A} \to \mathcal{A}$ which brings a single site representation into a double-site representation, a double-site into a triple-site, etc [53, 54]. Here, $\mathcal{A}$ is the Yangian of SU(2|3) . We forego our single index notation because this coproduct is dependent on the specifics of our generators and is easier to use with double indices.[1] The twisted coproduct for the ordinary generators is

$$\begin{aligned}
&\Delta R^a{}_b = K_{ab} \otimes R^a{}_b + R^a{}_b \otimes K_{ba}, \\
&\Delta L^\alpha{}_\beta = K_{\alpha\beta} \otimes L^\alpha{}_\beta + L^\alpha{}_\beta \otimes K_{\beta\alpha}, \\
&\Delta Q^c{}_\gamma = K_{c\gamma} \otimes Q^c{}_\gamma + Q^c{}_\gamma \otimes K_{\gamma c}, \\
&\Delta S^\gamma{}_c = K_{\gamma c} \otimes S^\gamma{}_c + S^\gamma{}_c \otimes K_{c\gamma}, \\
&\Delta D = 1 \otimes D + D \otimes 1.
\end{aligned} \tag{7.1}$$

As before, $1 \leq a, b \leq 3$ and $1 \leq \alpha, \beta \leq 2$, and now $1 \leq I, J, K \leq 5$. The twisted coproducts depend on the antisymmetric parameters $\alpha_{IJ} = -\alpha_{JI}$ which reside in $K_{IJ} =$

[1] A brief description can be found in Appendix B.

$\exp\left[\frac{i}{2}\sum_{K=1}^{5}\left(\alpha_{IK}-\alpha_{JK}\right)E_{KK}\right]$.[2] So,

$$
\begin{aligned}
K_{ab} &= e^{\frac{i}{2}\left(\alpha_{a\gamma}-\alpha_{b\gamma}\right)E_{\gamma\gamma}+\frac{i}{2}\left(\alpha_{ac}-\alpha_{bc}\right)E_{cc}} = K_{ba}^{-1}, \\
K_{\alpha\beta} &= e^{\frac{i}{2}\left(\alpha_{\alpha\gamma}-\alpha_{\beta\gamma}\right)E_{\gamma\gamma}+\frac{i}{2}\left(\alpha_{\alpha c}-\alpha_{\beta c}\right)E_{cc}} = K_{\beta\alpha}^{-1}, \\
K_{a\alpha} &= e^{\frac{i}{2}\left(\alpha_{a\gamma}-\alpha_{\alpha\gamma}\right)E_{\gamma\gamma}+\frac{i}{2}\left(\alpha_{ac}-\alpha_{\alpha c}\right)E_{cc}} = K_{\alpha a}^{-1}.
\end{aligned}
\tag{7.2}
$$

The quadratic Casimir of the SU(2|3) algebra with this twisted coproduct is

$$
g_{AB}\Delta J^A\Delta J^B = \frac{1}{3}\Delta D\Delta D + \frac{1}{2}\Delta L^{\gamma}{}_{\delta}\Delta L^{\delta}{}_{\gamma} - \frac{1}{2}\Delta R^{c}{}_{d}\Delta R^{d}{}_{c} - \frac{1}{2}\left[\Delta Q^{c}{}_{\gamma}, \Delta S^{\gamma}{}_{c}\right]. \tag{7.3}
$$

When we expand using the above coproducts, it can be shown that the phase contributions cancel in the single site components of the Casimir. As argued previously, these components give zero when acting on the states. Just as importantly, if we examine the cross terms we retrieve the deformed Hamiltonian discussed in the previous section. So, acting on a two-particle state $|\eta\rangle$,

$$
g_{AB}\Delta J^A\Delta J^B|\eta\rangle = \tilde{H}(1,2)|\eta\rangle, \tag{7.4}
$$

where $\tilde{H}(1,2)$ is given by (6.13) if we relate the deformation parameters with those from before, $\alpha_{IJ} = B_{IJ}$. So, the deformed Hamiltonian commutes with all of the ordinary symmetry generators for arbitrary B_{IJ}:

$$
[\tilde{H}(1,2), J^{A}{}_{B}(1,2)] = 0, \tag{7.5}
$$

[2]Twisted coproducts can be generated from a deforming function, F, such that $\Delta^{(F)} = F\Delta^{(0)}F^{-1}$, where $\Delta^{(0)}$ is the standard coproduct [31, 54]. The standard coproduct corresponds to $K_{IJ} = 1$.

where we construct $J^A{}_B(1,2)$ with the coproduct in (7.1). An example two-site generator is $R_{12}{}^a{}_b = R(1)^a{}_b K(2)_{ba} + K(1)_{ab} R(2)^a{}_b$. With this information we can begin to reconstruct the identities associated with the Yangian structure given in the previous section. To avoid confusion with the supercharge $Q^a{}_\alpha$ we shall denote the Yangian generator Q^A in double index notation, as $\hat{J}^A{}_B$. Coproducts for twisted Yangian generators [32, 55] of SU(2|3) take the form

$$
\begin{aligned}
\Delta\hat{R}^a{}_b &= K_{ab}\otimes\hat{R}^a{}_b + \hat{R}^a{}_b\otimes K_{ba} + \frac{1}{2}h\left(R^a{}_c K_{cb}\otimes K_{ca}R^c{}_b - K_{ac}R^c{}_b\otimes R^a{}_c K_{bc}\right)\\
&\quad + \frac{1}{2}h\left(Q^a{}_\gamma K_{\gamma b}\otimes K_{\gamma a}S^\gamma{}_b + K_{a\gamma}S^\gamma{}_b\otimes Q^a{}_\gamma K_{b\gamma}\right)\\
&\quad - \frac{1}{6}h\delta^a_b\left(Q^c{}_\gamma K_{\gamma c}\otimes K_{\gamma c}S^\gamma{}_c + K_{c\gamma}S^\gamma{}_c\otimes Q^c{}_\gamma K_{c\gamma}\right),\\
\Delta\hat{L}^\alpha{}_\beta &= K_{\alpha\beta}\otimes\hat{L}^\alpha{}_\beta + \hat{L}^\alpha{}_\beta\otimes K_{\beta\alpha} + \frac{1}{2}h\left(L^\alpha{}_\gamma K_{\gamma\beta}\otimes K_{\gamma\alpha}L^\gamma{}_\beta - K_{\alpha\gamma}L^\gamma{}_\beta\otimes L^\alpha{}_\gamma K_{\beta\gamma}\right)\\
&\quad + \frac{1}{2}h\left(S^\alpha K_{c\beta}\otimes K_{c\alpha}Q^c{}_\beta + K_{\alpha c}Q^c{}_\beta\otimes S^\alpha{}_c K_{\beta c}\right)\\
&\quad - \frac{1}{4}h\delta^\alpha_\beta\left(S^\gamma{}_c K_{c\gamma}\otimes K_{c\gamma}Q^c{}_\gamma + K_{\gamma c}Q^c{}_\gamma\otimes S^\gamma{}_c K_{\gamma c}\right),\\
\Delta\hat{Q}^a{}_\alpha &= K_{a\alpha}\otimes\hat{Q}^a{}_\alpha + \hat{Q}^a{}_\alpha\otimes K_{\alpha a} + \frac{1}{2}h\left(Q^a{}_\gamma K_{\gamma\alpha}\otimes K_{\gamma a}L^\gamma{}_\alpha - K_{a\gamma}L^\gamma{}_\alpha\otimes Q^a{}_\gamma K_{\alpha\gamma}\right)\\
&\quad + \frac{1}{2}h\left(R^a{}_c K_{c\alpha}\otimes K_{ca}Q^c{}_\alpha - K_{ac}Q^c{}_\alpha\otimes R^a{}_c K_{\alpha c}\right),\\
\Delta\hat{S}^\alpha{}_a &= K_{\alpha a}\otimes\hat{S}^\alpha{}_a + \hat{S}^\alpha{}_a\otimes K_{a\alpha} + \frac{1}{2}h\left(S^\alpha{}_c K_{ca}\otimes K_{ca}R^c{}_a - K_{\alpha c}R^c{}_a\otimes S^\alpha{}_c K_{ac}\right)\\
&\quad + \frac{1}{2}h\left(L^\alpha{}_\gamma K_{\gamma a}\otimes K_{\gamma\alpha}S^\gamma{}_a - K_{\alpha\gamma}S^\gamma{}_a\otimes L^\alpha{}_\gamma K_{a\gamma}\right),\\
\Delta\hat{D} &= 1\otimes\hat{D} + \hat{D}\otimes 1 + \frac{1}{4}h\left(S^\gamma{}_c K_{c\gamma}\otimes K_{c\gamma}Q^c{}_\gamma + K_{\gamma c}Q^c{}_\gamma\otimes S^\gamma{}_c K_{\gamma c}\right).
\end{aligned}
\tag{7.6}
$$

These coproducts are coassociative and quasi-cocommutative [56], and satisfy (4.6)-(4.8) in the double index basis. In the derivation of the above coproduct for the deformed SU(2|3) Yangians, we had to respect the even/odd property of the generators and the traceless condition of the even generators. The parameter h is related to α in the Serre relation (4.8), see appendix B.

We can use the twisted identity

$$\left[\tilde{H}_{12}, {q_{12}}^{A}{}_{B}\right] = 8h{\hat{J}_{12}}{}^{A}{}_{B}, \tag{7.7}$$

where ${q_{12}}^{A}{}_{B} = J^{A}{}_{B} \otimes K_{BA} - K_{AB} \otimes J^{A}{}_{B}$, and the $\hat{J}_{12}$ are given by the h dependent terms in (7.6). For example, the two-site tree level Yangian generator $\hat{Q}^{a}{}_{\alpha}$ is given by

$$\begin{aligned}{\hat{Q}_{12}}{}^{a}{}_{\alpha} = &\frac{1}{2}h\left(Q(1)^{a}{}_{\gamma}K(1)_{\gamma\alpha}K(2)_{\gamma a}L(2)^{\gamma}{}_{\alpha} - K(1)_{a\gamma}L(1)^{\gamma}{}_{\alpha}Q(2)^{a}{}_{\gamma}K(2)_{\alpha\gamma}\right)\\ &+\frac{1}{2}h\left(R(1)^{a}{}_{c}K(1)_{c\alpha}K(2)_{ca}Q(2)^{c}{}_{\alpha} - K(1)_{ac}Q(1)^{c}{}_{\alpha}R(2)^{a}{}_{c}K(2)_{\alpha c}\right).\end{aligned} \tag{7.8}$$

Then on two-sites we can show

$$\left[\tilde{H}_{12}, {\hat{J}_{12}}{}^{A}{}_{B}\right] = \tfrac{1}{2}h\, {q_{12}}^{A}{}_{B}, \tag{7.9}$$

acting on all the eigenstates.

In order to promote (7.5) and (7.9) to L sites, we construct the L-site representation for $J^{A}{}_{B}$ and $\hat{J}^{A}{}_{B}$ using twisted coproducts with (7.1) and (7.6). We find

$$\left[\tilde{H}, J^{A}{}_{B}\right] = 0, \tag{7.10}$$

and

$$\begin{aligned}\left[\tilde{H}, \hat{J}^{A}{}_{B}\right] &= \sum_{i=1}^{L-1}\left[\tilde{H}_{i,i+1}, {\hat{J}_{i,i+1}}{}^{A}{}_{B}\right]\\ &= \frac{1}{2}h\left(J(1)^{A}{}_{B}K(2)_{BA}\cdots K(L)_{BA} - K(1)_{AB}\cdots K(L-1)_{AB}J(L)^{A}{}_{B}\right)\end{aligned} \tag{7.11}$$

If we examine an infinite length chain, which would resemble the world-sheet of the dual string theory, we can assume that surface terms at infinity can be dropped [28], and in that sense, $[\tilde{H}, \hat{J}^A{}_B] = 0$. Thus, following the discussion in section 3, (7.11) provides a consistency check on the assumption that the SU(2|3) Yangian, with the twisted coproduct, holds to all orders in the Yang-Mills coupling constant.

Up to this point, the analysis in this section holds for arbitrary, antisymmetric α_{IJ}. We now illustrate the use of Yangians in these twisted theories in the two cases we examined earlier, in order to explain the residual symmetries.

7.1 Case 1: β Deformed $\gamma_1 = \gamma_2 = \gamma_3$

We examine the twisted coproducts of Case 1. Recall that the phase elements have the property $B_{a\alpha} = 0$, $B_{\alpha\beta} = 0$, and the B_{ab} sector contains some non-zero entries. We explicitly write the coproducts,

$$K_{ab} = e^{\frac{i}{2}(\alpha_{ac}-\alpha_{bc})E_{cc}} = K_{ba}^{-1}, \quad K_{\alpha\beta} = 1 = K_{\beta\alpha}^{-1}, \quad K_{a\alpha} = e^{\frac{i}{2}\alpha_{ac}E_{cc}} = K_{\alpha a}^{-1}. \tag{7.12}$$

We examine the symmetry after using the twisted coproducts and find that the residual SU(2)×U(1)^3 symmetry corresponds to an undeformed coproduct:

Residual Symmetries	**Broken Symmetries**
$\Delta L^\alpha{}_\beta = 1 \otimes L^\alpha{}_\beta + L^\alpha{}_\beta \otimes 1$	$\Delta R^a{}_b = K_{ab} \otimes R^a{}_b + R^a{}_b \otimes K_{ba}$
$\Delta D = 1 \otimes D + D \otimes 1$	$\Delta Q^c{}_\gamma = K_{c\gamma} \otimes Q^c{}_\gamma + Q^c{}_\gamma \otimes K_{\gamma c}$
$\Delta R^c{}_c = 1 \otimes R^c{}_c + R^c{}_c \otimes 1$	$\Delta S^\gamma{}_c = K_{\gamma c} \otimes S^\gamma{}_c + S^\gamma{}_c \otimes K_{c\gamma}$

Table 7.1: Residual and broken symmetries after β deformation.

Using these definitions one could check for the two-particle eigenstates $\widetilde{|\pm\rangle}$ listed in a previous section,

$$\left[\tilde{H}_{12}, \hat{J}_{12}{}^A{}_B\right] \widetilde{|\pm\rangle} = \tfrac{1}{2} h\, q_{12}{}^A{}_B \widetilde{|\pm\rangle}. \tag{7.13}$$

7.2 Case 2: δ Deformed $\gamma_1 = \gamma_2 = -\gamma_3$

We consider the richer structure of case 2. From previous sections we saw a residual SU(2|1) $\times$U(1)^2 symmetry. Recall that we have zero phase elements in the sectors $\alpha_{\alpha\beta} = \alpha_{3\alpha} = 0$. The other phases are $\alpha_{1\alpha} = -\alpha_{2\alpha}$ and $\alpha_{12} = \alpha_{13} = -\alpha_{23}$. In this section, we label the fields $\Phi_I = \{\phi_a, \phi_3, \psi_\alpha\}$, with the indices $1 \leq a, b \leq 2$ and $1 \leq \alpha, \beta \leq 2$. The twisted coproducts, have deformation parameters

$$
\begin{aligned}
&K_{\alpha\beta} = 1 = K^{-1}_{\beta\alpha}, \quad K_{3\alpha} = 1 = K_{\alpha 3}, \quad K_{33} = 1, \\
&K_{ab} = e^{\frac{i}{2}(\alpha_{a\gamma}-\alpha_{b\gamma})E_{\gamma\gamma}+\frac{i}{2}(\alpha_{a3}-\alpha_{b3})E_{33}+\frac{i}{2}(\alpha_{ac}-\alpha_{bc})E_{cc}} = K^{-1}_{ba}, \\
&K_{a3} = e^{\frac{i}{2}\alpha_{a\gamma}E_{\gamma\gamma}+\frac{i}{2}\alpha_{a3}E_{33}+\frac{i}{2}(\alpha_{ac}-\alpha_{3c})E_{33}} = K^{-1}_{3a}, \\
&K_{a\alpha} = e^{\frac{i}{2}\alpha_{a\gamma}E_{\gamma\gamma}+\frac{i}{2}(\alpha_{a3}-\alpha_{\alpha 3})E_{33}+\frac{i}{2}(\alpha_{ac}-\alpha_{\alpha c})E_{cc}} = K^{-1}_{\alpha a}.
\end{aligned} \tag{7.14}
$$

We apply these parameters and find the residual SU(2|1) $\times$U(1)^2 symmetry. Again,

Residual Symmetries	**Broken Symmetries**
$\Delta L^\alpha{}_\beta = 1 \otimes L^\alpha{}_\beta + L^\alpha{}_\beta \otimes 1$	$\Delta R^a{}_b = K_{ab} \otimes R^a{}_b + R^a{}_b \otimes K_{ba}$
$\Delta Q^3{}_\gamma = 1 \otimes Q^3{}_\gamma + Q^3{}_\gamma \otimes 1$	$\Delta Q^c{}_\gamma = K_{c\gamma} \otimes Q^c{}_\gamma + Q^c{}_\gamma \otimes K_{\gamma c}$
$\Delta S^\gamma{}_3 = 1 \otimes S^\gamma{}_3 + S^\gamma{}_3 \otimes 1$	$\Delta S^\gamma{}_c = K_{\gamma c} \otimes S^\gamma{}_c + S^\gamma{}_c \otimes K_{c\gamma}$
$\Delta D = 1 \otimes D + D \otimes 1$	
$\Delta R^c{}_c = 1 \otimes R^c{}_c + R^c{}_c \otimes 1$	

Table 7.2: Residual and broken symmetries after δ deformation.

one could directly compute, using the two-particle eigenstates in a previous section, to find $\left[\tilde{H}_2, \hat{J}^A{}_B\right]\widetilde{|\pm\rangle} = q_{12}{}^A{}_B\widetilde{|\pm\rangle}$.

In both cases, since the coproducts for the remaining symmetries are non-standard and contain deformation parameters, these signal broken symmetries in the corresponding deformed gauge field theories.

Chapter 8

Conformal Supergravity Tree Amplitudes from Open Twistor Strings

Twistor space in string theory has received much interest in recent years. In 2004 two equivalent twistor string models emerged. Witten proposed a topological B model string theory with target space given by the supermanifold $\mathbb{C}\mathrm{P}^{3|4}$. Berkovits proposed an open string version with a first order world-sheet action. The target space of these theories is related to Penrose's twistor space $\mathbb{C}\mathrm{P}^3$ [64]. In four-dimensions we go from vector to spinor indices as

$$x_{a\dot{a}} \equiv x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}, \tag{8.1}$$

where $\sigma_\mu = (1, \sigma^i)$. We can retrieve space time coordinates by $x^\mu = \frac{1}{2}\mathrm{Tr}\left[\sigma^\mu_{\alpha\dot{\alpha}} x^{\alpha\dot{\alpha}}\right]$. The space time coordinates are exchanged for two two-spinors $(\pi^a,\, \omega_{\dot{a}})$,

$$\pi^a = x^{a\dot{a}}\,\omega_{\dot{a}} = x^\mu \sigma_\mu^{a\dot{a}} \omega_{\dot{a}} \tag{8.2}$$

These spinors define complex two-planes in complexified Minkowski space. The wave function describing states with the position π, ω in twistor space includes the delta functions $\delta(\lambda^a(\rho) - \pi^a)$ and $\delta(\mu^{\dot{a}}(\rho) - \omega^{\dot{a}})$. We do a Fourier transform on ω in the second delta function, and express it as the exponential, $\exp[i\mu^{\dot{a}}(\rho)\bar{\pi}_{\dot{a}}]$, to arrive at the canonical vertex operators, presented in the next chapter.

We can see from (8.1) that $(p^0)^2 - (\vec{p})^2 = \det p^{a\dot{a}}$. For massless particles we can write

$$p^{a\dot{a}} = \pi^a \bar{\pi}^{\dot{a}}. \tag{8.3}$$

If we have a second vector $q^{a\dot{a}} = \mu^a \bar{\mu}^{\dot{a}}$, the product of the two vectors p, q becomes $2\, p \cdot q = \epsilon^{ab}\epsilon^{\dot{a}\dot{b}} p_{a\dot{a}} q_{b\dot{b}} = \epsilon^{ab}\epsilon^{\dot{a}\dot{b}} \lambda_a \bar{\lambda}_{\dot{a}} \mu_b \bar{\mu}_{\dot{b}} = \langle \lambda\, \mu \rangle\, [\bar{\lambda}\, \bar{\mu}]$, where $\epsilon^{12} = 1 = -\epsilon^{21} = -\epsilon_{12} = \epsilon_{21}$.

The twistor string describes a perturbative $\mathcal{N} = 4$ super Yang-Mills theory. Superconformal invariance is manifest and therefore any resulting gravity will be conformal supergravity. This four-dimensional $\mathcal{N} = 4$ conformally invariant supergravity theory coupled to $\mathcal{N} = 4$ super Yang-Mills is a finite theory [37]-[39].

The world-sheet fields for twistor string theory are the twistor fields Y_I, Z^I, $1 \leq I \leq 8$, and the current algebra J^A with central charge 28, where A runs over the dimension of the gauge symmetry group. In addition there are ghost fields b, c, u, v, and world sheet gauge fields all summarized in [51]. The homogenous coordinates in twistor space are the fields Z^I. They have conformal spin zero and are labeled as $Z^I = \{\lambda^a, \mu^{\dot{a}}, \psi^m\}$, where the four boson fields are $\lambda^a, \mu^{\dot{a}}$, $1 \leq a, \dot{a} \leq 2$ and the four fermion fields are ψ^m, $1 \leq m \leq 4$. The conjugate variables Y_I have conformal spin one. We expand the fields in modes with the form

$$\Psi(\rho) = \sum_n \Psi_n \rho^{-n-\mathcal{J}}, \tag{8.4}$$

where $\mathcal{J}$ is the conformal spin. Therefore, the mode expansion of the twistor fields is

$$Z^I(\rho) = \sum_n Z_n \, \rho^{-n}, \quad \text{and} \quad Y_I(\rho) = \sum_n Y_{I\,n} \, \rho^{-n-1}. \tag{8.5}$$

The nonzero twistor field commutation relations follow from

$$Z^I(\rho) Y_J(\zeta) =: Z^I(\rho) Y_J(\zeta) : + \delta^I_J (\rho - \zeta)^{-1}. \tag{8.6}$$

In the remaining portion of this dissertation, we concentrate on deriving vertex operators for the supergravitons and gluons. We then calculate n-point tree amplitudes, including calculations involving 'dipole' graviton states. These states are derived from the fourth order differential equation for the metric field and are not eigenstates of the translation generator.

Chapter 9

Vertex Operators and Canonical Quantization

9.1 Vertex Operators

The massless states of $\mathcal{N}=4$ conformal supergravity consist of pairs of graviton supermultiplets (called dipoles), whose vertex operators are $V_F(\rho)$, $V_{F'}(\rho)$ and $V_G(\rho), V_{G'}(\rho)$; in addition to spin 3/2 supermultiplets, with vertex operators $V_f(\rho)$ and $V_g(\rho)$. Loosely following the notation of [35], we list them in terms of homogeneous functions f^I, g_I, of Z^I in Table 9.1. For each vertex, f^I and g_I satisfy

$$\frac{\partial}{\partial Z^I} f^I = 0, \quad Z^I g_I = 0 \tag{9.1}$$

to ensure the vertex operators are primary with respect to the $U(1)$ current

$$J(\rho) = -\sum_I : Y_I(\rho) Z^I(\rho) : \tag{9.2}$$

and the Virasoro current

$$L(\rho) = -\sum_I : Y_I(\rho) Z^I(\rho) : - : u(\rho)v(\rho) : +2 : \partial c(\rho) b(\rho) : - : \partial b(\rho) c(\rho) : + L^J(\rho). \tag{9.3}$$

Here $L^J(\rho)$ is the contribution from the current algebra. The vertex operators have charge zero and conformal dimension one. The primed vertices correspond to states that do not diagonalize the translation generators [35].

Vertex Operator	**Helicities**
$V_F(\rho) = f^{\dot{a}}(Z(\rho))Y_{\dot{a}}(\rho)$	$(2,\ \frac{3}{2},\ 1,\ \frac{1}{2},\ 0)$
$V_G(\rho) = g_a(Z(\rho))\partial\lambda^a(\rho)$	$(0,\ -\frac{1}{2},\ -1,\ -\frac{3}{2},\ -2)$
$V_{F'}(\rho) = f^a(Z(\rho))Y_a(\rho) + \hat{f}^{\dot{a}}(Z(\rho))Y_{\dot{a}}(\rho)$	$(2,\ \frac{3}{2},\ 1,\ \frac{1}{2},\ 0)$
$V_{G'}(\rho) = g_{\dot{a}}(Z(\rho))\partial\mu^{\dot{a}}(\rho) + \hat{g}_a(Z(\rho))\partial\lambda^a(\rho)$	$(0,\ -\frac{1}{2},\ -1,\ -\frac{3}{2},\ -2)$
$V_f(\rho) = f^m(Z(\rho))Y_m(\rho) + \tilde{f}^{\dot{a}}(Z(\rho))Y_{\dot{a}}(\rho)$	$(\frac{3}{2},\ 1,\ \frac{1}{2},\ 0,\ -\frac{1}{2})$
$V_g(\rho) = g_m(Z(\rho))\partial\psi^m(\rho) + \tilde{g}_a(Z(\rho))\partial\lambda^a(\rho)$	$(\frac{1}{2},\ 0,\ -\frac{1}{2},\ -1,\ -\frac{3}{2})$
$V_\Phi^A(\rho) = V_\phi(Z(\rho))J^A(\rho)$	$(\pm 1,\ 4(\pm\frac{1}{2}),\ 6(0))$

Table 9.1: Vertex operators and helicities for $\mathcal{N} = 4$ conformal supergravity theory

We will define the homogeneous functions for each vertex operator, and discuss their properties. The states are labeled by helicities and their representations under the $SU(4)$ $R-$symmetry (in bold). As a reminder, we first look at the $\mathcal{N} = 4$ Yang-Mills gluon vertex,

$$V_\Phi^A(\rho) = V_\phi(Z(\rho))J^A(\rho) \tag{9.4}$$

with

$$
\begin{aligned}
V_\phi(Z(\rho)) &= \int \frac{dk}{k} \prod_{a=1}^{2} \delta(k\lambda^a(\rho) - \pi^a) e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)} \\
&\quad \times \left[A_1 + k\psi^b A_b + \frac{k^2}{2}\psi^b\psi^c A_{bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d A_{bcd} + k^4\psi^1\psi^2\psi^3\psi^4 A_{-1} \right] \\
&= \frac{1}{(\pi^1)^2}\delta\left(\frac{\lambda^2(\rho)}{\lambda^1(\rho)} - \frac{\pi^2}{\pi^1}\right) \exp\left\{ i\frac{\mu^{\dot{b}}(\rho)\bar{\pi}_{\dot{b}}\pi^1}{\lambda^1(\rho)} \right\} \\
&\quad \times \left[A_+ + \frac{\pi^1}{\lambda^1(\rho)}\psi^b A_b + \left(\frac{\pi^1}{\lambda^1(\rho)}\right)^2 \frac{1}{2}\psi^b\psi^c A_{bc} \right. \\
&\quad\quad \left. + \left(\frac{\pi^1}{\lambda^1(\rho)}\right)^3 \frac{1}{3!}\psi^b\psi^c\psi^d A_{bcd} + \left(\frac{\pi^1}{\lambda^1(\rho)}\right)^4 \psi^1\psi^2\psi^3\psi^4 A_- \right]
\end{aligned} \tag{9.5}
$$

where $\psi^b \equiv \psi^b(\rho)$ and b, c, d are summed over. With use of the delta function $\delta(k\lambda^1(\rho) - \pi^1)$ to perform the k-integration, this becomes the vertex used by Berkovits and Witten, except that they omit the A_b, A_{bc}, and A_{bcd} terms [51, 34, 58, 35]. In that form, it is easy to see that the vertex operator $V_\phi(Z^I(\rho))$ is homogeneous in $Z^I(\rho)$ of degree $p = 0$. (A function homogeneous in Z of degree p satisfies $f(kZ) = k^p f(Z)$, so it has $U(1)$ charge p.) For the scaling $\pi^a \to \kappa\pi^a$, $\bar{\pi}^a \to \kappa^{-1}\bar{\pi}^a$, each helicity component scales as κ^{-2h} where h is the helicity of the state in Minkowski spacetime [35]. Thus $V_\phi(Z(\rho))$ describes the super gluon helicity states $(1, \mathbf{1}), (\frac{1}{2}, \bar{\mathbf{4}}), (0, 6), (-\frac{1}{2}, \mathbf{4}), (-1, \mathbf{1})$. The spinor helicity variables $\pi^a, \bar{\pi}^{\dot{a}}$ are related to massless four-dimensional momentum $p_{a\dot{a}} = \pi_a\bar{\pi}_{\dot{a}}$.

F Vertices

For the conformal supergravity states, the vertex operator for the helicity states $(2, \mathbf{1}), (\frac{3}{2}, \bar{\mathbf{4}}), (1, \mathbf{6}), (\frac{1}{2}, \mathbf{4}), (0, \mathbf{1})$ is given by

$$
V_F(\rho) = f^{\dot{a}}(Z(\rho))\, Y_{\dot{a}}(\rho) \tag{9.6}
$$

with

$$f^{\dot{a}}(Z(\rho)) = i\int \frac{dk}{k^2}\bar{\pi}^{\dot{a}}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}$$
$$\times\left[e_2 + k\psi^b\eta_{\frac{3}{2}b} + \frac{k^2}{2}\psi^b\psi^c T_{1bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d\Lambda_{\frac{1}{2}bcd} + k^4\psi^1\psi^2\psi^3\psi^4\bar{C}_0\right] \quad (9.7)$$

The function $f^{\dot{a}}(Z^I(\rho))$ is homogeneous in $Z^I(\rho)$ of degree 1. The highest component (which is proportional to e_2) scales as κ^{-4} with π^a and $\bar{\pi}^a$, to describe helicity 2. As required by the primary field conditions, $\frac{\partial}{\partial\mu^{\dot{a}}(\rho)}f^{\dot{a}}(Z(\rho)) = 0$, since $\bar{\pi}_{\dot{a}}\bar{\pi}^{\dot{a}} = 0$. These vertices correspond to plane wave states and diagonalize the translation generators. Together with the F' vertices they comprise a dipole pair [35].

F' Vertices

The vertex operator for a second set of states $(2,\mathbf{1}),(\frac{3}{2},\bar{\mathbf{4}}),(1,\mathbf{6}),(\frac{1}{2},\mathbf{4}),(0,\mathbf{1})$ is

$$V_{F'}(\rho) =:f^a(Z(\rho))\,Y_a(\rho): + :\hat{f}^{\dot{a}}(Z(\rho))\,Y_{\dot{a}}(\rho): \quad (9.8)$$

with

$$f^a(Z(\rho)) = \bar{s}^a\int \frac{dk}{k^2}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}$$
$$\times\left[e_2' + k\psi^b\eta'_{\frac{3}{2}b} + \frac{k^2}{2}\psi^b\psi^c T'_{1bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d\Lambda'_{\frac{1}{2}bcd} + k^4\psi^1\psi^2\psi^3\psi^4\bar{C}_0'\right] \quad (9.9)$$

and

$$\hat{f}^{\dot{a}}(Z(\rho)) = -is^{\dot{a}}\bar{s}^e\int \frac{dk}{k^3}\frac{\partial}{\partial\lambda^e(\rho)}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}$$
$$\times\left[e_2' + k\psi^b\eta'_{\frac{3}{2}b} + \frac{k^2}{2}\psi^b\psi^c T'_{1bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d\Lambda'_{\frac{1}{2}bcd} + k^4\psi^1\psi^2\psi^3\psi^4\bar{C}_0'\right] \quad (9.10)$$

chosen to satisfy the volume preserving condition $\frac{\partial}{\partial\lambda^a(\rho)} f^a(Z(\rho)) + \frac{\partial}{\partial\mu^{\dot a}(\rho)} \hat f^{\dot a}(Z(\rho)) = 0$. The spinors $s_{\dot a}$ and $\bar s_a$ are defined such that $\pi^a \bar s_a = 1$ and $\bar\pi^{\dot a} s_{\dot a} = 1$. These states are not eigenstates of the momentum operator, as we discuss in (10.45).

G Vertices

Conformal supergravity states with the opposite helicities and conjugate $SU(4)$ representations, $(0, \mathbf{1}), (-\frac{1}{2}, \bar{\mathbf{4}}), (-1, \mathbf{6}), (-\frac{3}{2}, \mathbf{4}), (-2, \mathbf{1})$ are described by

$$V_G(\rho) = g_a(Z(\rho))\, \partial\lambda^a(\rho) \tag{9.11}$$

with

$$g_a(Z(\rho)) = \int dk\, k\, \lambda_a(\rho) \prod_{a=1}^{2} \delta(k\lambda^a(\rho) - \pi^a) e^{ik\bar\pi_{\dot b}\mu^{\dot b}(\rho)}$$
$$\times \left[C_0 + k\psi^b \Lambda_{-\frac{1}{2}b} + \frac{k^2}{2}\psi^b\psi^c T_{-1bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d \eta_{-\frac{3}{2}bcd} + k^4\psi^1\psi^2\psi^3\psi^4 e_{-2} \right] \tag{9.12}$$

$g_a(Z^J(\rho))$ is homogeneous in $Z^J(\rho)$ of degree -1. The highest component (proportional to C) scales with π^a and $\bar\pi^a$ as κ^0 for zero helicity. Also, $\lambda^a(\rho) g_a(Z(\rho)) = 0$. These are momentum eigenstates, and form a dipole pair with the G' vertices.

G' Vertices

The final states that do not diagonalize the translation generators form a second set of states $(0, \mathbf{1}), (-\frac{1}{2}, \bar{\mathbf{4}}), (-1, \mathbf{6}), (-\frac{3}{2}, \mathbf{4}), (-2, \mathbf{1})$ and correspond to

$$V_{G'}(\rho) = g_{\dot a}(Z(\rho))\, \partial\mu^{\dot a}(\rho) + \hat g_a(Z(\rho))\, \partial\lambda^a(\rho) \tag{9.13}$$

with

$$g_{\dot{a}}(Z(\rho)) = is_{\dot{a}} \int dk \prod_{a=1}^{2} \delta(k\lambda^a(\rho) - \pi^a) e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}$$
$$\times \left[C_0' + k\psi^b \Lambda'_{-\frac{1}{2}b} + \frac{k^2}{2}\psi^b\psi^c T'_{-1bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d \eta'_{-\frac{3}{2}bcd} + k^4\psi^1\psi^2\psi^3\psi^4 e'_{-2} \right] \tag{9.14}$$

and

$$\hat{g}_a(Z(\rho)) = -i\bar{s}_a s_{\dot{a}} \mu^{\dot{a}}(\rho) \int dk\, k \prod_{a=1}^{2} \delta(k\lambda^a(\rho) - \pi^a) e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}$$
$$\times \left[C_0' + k\psi^b \Lambda'_{-\frac{1}{2}b} + \frac{k^2}{2}\psi^b\psi^c T'_{-1bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d \eta'_{-\frac{3}{2}bcd} + k^4\psi^1\psi^2\psi^3\psi^4 e'_{-2} \right] \tag{9.15}$$

with $\mu^{\dot{a}}(\rho) g_{\dot{a}}(Z(\rho)) + \lambda^a(\rho)\hat{g}_a(Z(\rho)) = 0$.

f Vertices

The vertex operator for the plane wave states with quantum numbers $(\frac{3}{2}, \mathbf{4})$, $(1, \mathbf{15} \oplus \mathbf{1})$, $(\frac{1}{2}, \overline{\mathbf{20}} \oplus \bar{\mathbf{4}})$, $(0, \mathbf{10} \oplus \mathbf{6})$, $(-\frac{1}{2}, \mathbf{4})$ is

$$V_f(\rho) = f^m\,(Z(\rho))\, Y_m(\rho) \; + \; \tilde{f}^{\dot{a}}\,(Z(\rho))\, Y_{\dot{a}}(\rho) \tag{9.16}$$

with

$$f^m(Z(\rho)) = \int \frac{dk}{k^2} \prod_{a=1}^{2} \delta(k\lambda^a(\rho) - \pi^a) e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}$$
$$\times \left[E^m_{\frac{3}{2}} + k\psi^b E^m_{1b} + \frac{k^2}{2}\psi^b\psi^c E^m_{\frac{1}{2}bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d E^m_{0bcd} + k^4\psi^1\psi^2\psi^3\psi^4 E^m_{-\frac{1}{2}} \right] \tag{9.17}$$

and

$$\tilde{f}^{\dot{a}}(Z(\rho)) = -is^{\dot{a}} \int \frac{dk}{k^2} \prod_{a=1}^{2} \delta(k\lambda^a(\rho) - \pi^a) e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}$$
$$\times \left[E^m_{1m} + k\psi^c E^m_{\frac{1}{2}mc} + \frac{k^2}{2}\psi^c\psi^d E^m_{0mcd} + \frac{k^3}{3!}\psi^b\psi^c\psi^d \epsilon_{mbcd} E^m_{-\frac{1}{2}} \right] \quad (9.18)$$

so that $\frac{\partial}{\partial\psi^m(\rho)} f^m(Z(\rho)) + \frac{\partial}{\partial\mu^{\dot{a}}(\rho)} \tilde{f}^{\dot{a}}(Z(\rho)) = 0$, $f^m(Z(\rho)$ and $\tilde{f}^{\dot{a}}(Z(\rho))$ have degree 1, and the leading components scale as κ^{-3} and κ^{-2} respectively.

g Vertices

The vertex operator for states with the opposite helicities and conjugate $SU(4)$ representations, $(\frac{1}{2}, \bar{\mathbf{4}})$, $(0, \overline{\mathbf{10}} \oplus \mathbf{6})$, $(-\frac{1}{2}, \mathbf{20} \oplus \mathbf{4})$, $(-1, \mathbf{1} \oplus \mathbf{15})$, $(-\frac{3}{2}, \bar{\mathbf{4}})$ is

$$V_g(\rho) = g_m\left(Z(\rho)\right)\partial\psi^m(\rho) \; + \; \tilde{g}_a\left(Z(\rho)\right)\partial\lambda^a(\rho) \quad (9.19)$$

with

$$g_m(Z(\rho)) = \int dk \prod_{a=1}^{2} \delta(k\lambda^a(\rho) - \pi^a) e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)} \left[\bar{E}_{\frac{1}{2}m} + k\psi^b \bar{E}_{0mb} \right.$$
$$\left. + \frac{k^2}{2}\psi^b\psi^c \bar{E}_{-\frac{1}{2}mbc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d \bar{E}_{-1mbcd} + k^4\psi^1\psi^2\psi^3\psi^4 \bar{E}_{-\frac{3}{2}m} \right] \quad (9.20)$$

and

$$\tilde{g}_a(Z(\rho)) = \bar{s}_a \int dk\, k \prod_{a=1}^{2} \delta(k\lambda^a(\rho) - \pi^a) e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}$$
$$\times \psi^m \left[\bar{E}_{\frac{1}{2}m} + k\psi^b \bar{E}_{0mb} + \frac{k^2}{2}\psi^b\psi^c \bar{E}_{-\frac{1}{2}mbc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d \bar{E}_{-1mbcd} \right] \quad (9.21)$$

where $\psi^m(\rho)g_m(Z(\rho)) + \lambda^a(\rho)\tilde{g}_a(Z(\rho)) = 0$. To obtain $\tilde{g}_a(Z(\rho))$, we use $\pi^a\bar{s}_a = 1$ which can be written as $\frac{\pi 1}{\lambda^1(\rho)}\lambda^a(\rho)\bar{s}_a = 1$ on the support of the delta function $\delta\left(\frac{\lambda^2(\rho)}{\lambda^1(\rho)} - \frac{\pi^2}{\pi^1}\right)$.

9.2 Norm of the States

We can check that the norms of the one-particle graviton and scalar states are zero. The physical states corresponding to the vertex operators are shown in Table 9.2, where

V_F	$f^{\dot{a}}(Z_0)Y_{\dot{a}(-1)}\vert 0\rangle$
V_G	$g_a(Z_0)\lambda^a_{(-1)}\vert 0\rangle$
$V_{F'}$	$\left(f^a(Z_0)Y_{a(-1)} + \hat{f}^{\dot{a}}(Z_0)Y_{\dot{a}(-1)}\right)\vert 0\rangle$
$V_{G'}$	$\left(g_{\dot{a}}(Z_0)\mu^{\dot{a}}_{(-1)} + \hat{g}_a(Z_0)\lambda^a_{(-1)}\right)\vert 0\rangle$
V_f	$\left(f^m(Z_0)Y_{m(-1)} + \tilde{f}^{\dot{a}}(Z_0)Y_{\dot{a}(-1)}\right)\vert 0\rangle$
V_g	$\left(g_m(Z_0)\psi^m_{(-1)} + \tilde{g}_a(Z_0)\lambda^a_{(-1)}\right)\vert 0\rangle$
V^A_Φ	$V_\phi(Z_0)J^A_{(-1)}\vert 0\rangle$

Table 9.2: One particle states

the mode expansion for the twistor fields is $Z^I(\rho) = \sum_n Z^I_n \rho^{-n},\ Y_J(\rho) = \sum_n Y_{Jn}\rho^{-n-1}$, and the modes annihilating the vacuum are $Z^I_n|0\rangle = 0,\ n \geq 1$, and $Y_{nI}|0\rangle = 0,\ n \geq 0$.

The canonical commutation relations are

$$\left[Z^I_n, Y_{Jm}\right] = \delta^I_J\,\delta_{n,-m}, \tag{9.22}$$

and the hermitian conjugates [51] are $(Z^I_n)^\dagger = Z^I_{-n}$, for $1 \leq I \leq 8$; and $(Y^J_n)^\dagger = -Y^J_{-n}$, for $1 \leq J \leq 4$; and $(Y^J_n)^\dagger = Y^J_{-n}$, for $5 \leq J \leq 8$.

We compute the norms as follows. For example, for V_F,

$$||f^{\dot{a}}(Z_0)Y_{\dot{a}\,(-1)}|0\rangle|| = -\langle 0|Y_{\dot{b}\,(1)}(f^{\dot{b}}(Z_0))^* f^{\dot{a}}(Z_0)Y_{\dot{a}\,(-1)}|0\rangle = 0 \tag{9.23}$$

since Z^I_0 and $Y_{J(-1)}$ commute, and the $Y_{J(-1)}$ acting to the left annihilate the vacuum.

For $V_{F'}$,

$$
\begin{aligned}
&||(f^a(Z_0)Y_{a(-1)} + \hat{f}^{\dot{a}}(Z_0)Y_{\dot{a}(-1)})|0\rangle|| \\
&\quad = \langle 0|(Y_{\dot{b}(1)}(\hat{f}^{\dot{b}}(Z_0))^* + Y_{b(1)}(f^b(Z_0)^*)(f^a(Z_0)Y_{a(-1)} + \hat{f}^{\dot{a}}(Z_0)Y_{\dot{a}(-1)})|0\rangle = 0. \quad (9.24)
\end{aligned}
$$

Similarly, the norms for the states $V_G, V_{G'}, V_f, V_g$ all vanish, since they involve different Y and Z modes and thus commute, allowing the negative Y modes to annihilate the left vacuum. In contrast, the gluon norm is positive,

$$
\begin{aligned}
||V_\phi(Z_0)J^A_{-1}|0\rangle|| &= \langle 0|J^A_1 V^*_\phi(Z_0)V_\phi(Z_0)J^A_{-1}|0\rangle \\
&= \langle 0|J^A_1 J^A_{-1}|0\rangle \int dZ_0|V_\phi(Z_0)|^2 = k\int dZ_0|V_\phi(Z_0)|^2 > 0, \quad (9.25)
\end{aligned}
$$

where k is the level of the current algebra, $J^A_n J^B_m = if^{AB}{}_C J^C_{n+m} + kn\delta_{n,-m}\delta^{AB}$.

Chapter 10

Three Point Couplings

In this section, we compute non-vanishing three-point amplitudes for the gravitons, scalars, and gluons in the V_F, V_G and V_Φ vertices using canonical quantization, and then extend these to the corresponding states in the primed vertices $V_{F'}$, and $V_{G'}$. We have relabeled this subset of vertex operators for positive and negative helicity states in Table C.1. (It will be convenient to consider the scalars $\bar{C}, \bar{C}'$ as negative helicity, and C, C' as positive helicity, when computing amplitudes, as in [35].) Amplitudes for other states can be calculated with similar ease.

Scattering amplitudes in twistor string theory receive contributions from the various instanton sectors, which are due to world sheet gauge fields [33, 34]. Amplitudes with the number of negative helicity states equal to $d + 1 - \ell$ are computed with instanton number d, where ℓ is the number of loops. For tree amplitudes, $\ell = 0$. We compute the N-point tree as [51]

$$\langle V_1(\rho_1)V_2(\rho_2)\dots V_N(\rho_N)\rangle_{\rm tree} = \int \langle 0|e^{dq_0}V_1(\rho_1)V_2(\rho_2)\cdots V_N(\rho_N)|0\rangle \prod_{r=1}^{N} d\rho_r/d\gamma_M d\gamma_S \tag{10.1}$$

where $d\gamma_M$ is the invariant measure of the Mobius group, and $d\gamma_S$ is the invariant measure of the scaling group. q_0 is the conjugate zero mode of the $U(1)$ current and commutes with field modes as $Y^I_{n-d}e^{dq_0} = e^{dq_0}Y^I_n$ and $Z^I_{n+d}e^{dq_0} = e^{dq_0}Z^I_n$.

10.1 Unprimed Couplings

Using the canonical methods of [51], we compute the non-vanishing three-point tree amplitudes that come from the degree one curves as follows.

$$\begin{aligned}
\langle A^{A_1}_{-1}(\rho_1)A^{A_2}_{-1}(\rho_2)C(\rho_3)\rangle_{\rm tree} &= \int \langle 0|e^{q_0}A^{A_1}_{-1}(\rho_1)A^{A_2}_{-1}(\rho_2)C(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\
&= \int \prod_{r=1}^{3} dk_r \prod_{r=1}^{3} k_r\lambda_a(\rho_3)\partial\lambda^a(\rho_3) \prod_{r,a}\delta({\pi_r}^a - k_r\lambda^a(\rho_r))(\rho_1-\rho_2)^4(k_1k_2)^2 \\
&\quad\times \langle e^{i\sum_{r=1}^{3}k_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}\rangle \prod_a d^2\lambda^a \prod_r d\rho_r/d\gamma_M d\gamma_S \\
&\quad\times \langle J^{A_1}(\rho_1)J^{A_2}(\rho_2)\rangle\, A_{-1(1)}A_{-1(2)}C_{0(3)},
\end{aligned} \tag{10.2}$$

where we find the expectation of the current to be [59]

$$\langle J^{A_1}(\rho_1)J^{A_2}(\rho_2)\rangle = \frac{\delta^{A_1A_2}}{(\rho_1-\rho_2)(\rho_2-\rho_1)}. \tag{10.3}$$

In $d=1$, we replace $Z^I(\rho)$ with $Z^I_0 + \rho Z^I_{-1}$, and we find

$$\lambda_a(\rho)\,\partial\lambda^a(\rho) = (\det\lambda), \tag{10.4}$$

and

$$\langle 0|e^{q_0}e^{i\sum_{r=1}^{3}k_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}|0\rangle = \delta^2(\Sigma_r k_r\bar{\pi}_{r\dot{b}})\delta^2(\Sigma_r\rho_r k_r\bar{\pi}_{r\dot{b}}), \tag{10.5}$$

with $(\det\lambda) = \lambda^1_0\lambda^2_{-1} - \lambda^2_0\lambda^1_{-1}$. Next we integrate over k_r using the delta functions $\delta({\pi_r}^1 - k_r\lambda^1(\rho_r))$. We then use the identity

$$\delta^2\left(\sum_r \frac{{\pi_r}^1\bar{\pi}_{r\dot{b}}}{\lambda^1(\rho_r)}\right)\delta^2\left(\sum_r \rho_r\frac{{\pi_r}^1\bar{\pi}_{r\dot{b}}}{\lambda^1(\rho_r)}\right) = (\det\lambda)^2\,\delta^4(\Sigma_r{\pi_r}^a\bar{\pi}_{r\dot{b}}). \tag{10.6}$$

For the remainder of this chapter, we use the notation $\delta^4(\Sigma_r {\pi_r}^a \bar{\pi}_{r\dot{b}}) = \delta^4(\Sigma_r \pi_r \bar{\pi}_r)$. The amplitude is now

$$\begin{aligned}
&\langle A^{A_1}_{-1}(\rho_1) A^{A_2}_{-1}(\rho_2) C(\rho_3)\rangle_{\text{tree}} \\
&\quad = \int \prod_r \frac{{\pi_r}^1}{(\lambda^1(\rho_r))^2} \prod_r \delta\left({\pi_r}^2 - \frac{\lambda^2(\rho_r)}{\lambda^1(\rho_r)}{\pi_r}^1\right) (\rho_1-\rho_2)^2 \left(\frac{{\pi_1}^1 {\pi_2}^1}{\lambda^1(\rho_1)\lambda^1(\rho_2)}\right)^2 (\det\lambda)^3 \\
&\qquad \times\ \delta^4(\Sigma_r \pi_r \bar{\pi}_r) \prod_a d^2\lambda^a \prod_r d\rho_r / d\gamma_M d\gamma_S \left[-\delta^{A_1 A_2} A_{-1(1)} A_{-1(2)} C_{0(3)}\right]. \qquad (10.7)
\end{aligned}$$

The invariant measures of the above amplitude $d\gamma_M$ and $d\gamma_S$ are the Mobius invariance of $\mathrm{SL}(2,\mathbb{R})$ and a scaling invariance. As the above amplitude is dependent only on the twistor fields $\lambda^a(\rho)$ the scale invariance manifests itself as

$$\lambda^a(\rho) \longrightarrow \lambda^{a\prime}(\rho) = \alpha\lambda^a(\rho), \qquad (10.8)$$

where α is some constant. It is easy to see that under this transformation

$$\prod_{a=1}^{2} d^2\lambda^a \longrightarrow \prod_{a=1}^{2} d^2\lambda^{a\prime} = \alpha^4 \prod_{a=1}^{2} d^2\lambda^a, \qquad (10.9)$$

and

$$\det\lambda \longrightarrow \det\lambda' = \alpha^2\ \det\lambda. \qquad (10.10)$$

Careful counting of α gives ten inverse factors of α from the inverse $\lambda^1(\rho)$'s, six factors from $(\det\lambda)^3$, and four factors from $d^2\lambda^a$. And as all the factors cancel the above is scale invariant.

The above amplitude is also $\mathrm{SL}(2,\mathbb{R})$ invariant, that is

$$\rho \longrightarrow \rho' = \frac{a\,\rho + b}{c\,\rho + d}, \qquad (10.11)$$

with $a, b, c, d \in \mathbb{R}$ and $a\,d - b\,c \neq 0$. Under this transformation, the measure

$$d\rho' = \frac{(a\,d - b\,c)}{(c\,\rho + d)^2}\, d\rho, \tag{10.12}$$

and

$$\rho_i' - \rho_j' = \frac{(a\,d - b\,c)}{(c\,\rho_i + d)(c\,\rho_j + d)}\,(\rho_i - \rho_j)\,. \tag{10.13}$$

The twistor fields $\lambda^a(\rho)$ transform as $\lambda^a(\rho) \to \lambda^a(\rho')$. We express these fields $\lambda^a(\rho')$ in terms of a new twistor field $\tilde{\lambda}^a$ with the old variable ρ

$$\begin{aligned}
\lambda^a(\rho) \longrightarrow \lambda^a(\rho') &= \lambda_0^a + \rho' \lambda_{-1}^a \\
&= \lambda_0^a + \left(\frac{a\,\rho + b}{c\,\rho + d}\right) \lambda_{-1}^a \\
&= \frac{1}{c\,\rho + d}[(d\lambda_0^a + b\lambda_{-1}^a) + \rho(c\lambda_0^a + a\lambda_{-1}^a)] \\
&= \frac{1}{c\,\rho + d}[\tilde{\lambda}_0^a + \rho\,\tilde{\lambda}_{-1}^a] = \frac{1}{c\,\rho + d}\tilde{\lambda}^a(\rho).
\end{aligned} \tag{10.14}$$

Where we have defined $\tilde{\lambda}_0^a = d\lambda_0^a + b\lambda_{-1}^a$ and $\tilde{\lambda}_{-1}^a = c\lambda_0^a + a\lambda_{-1}^a$. We express $\det \lambda$ in terms of the new fields

$$\det \lambda = \frac{1}{(a\,d - b\,c)^2} \det \tilde{\lambda}. \tag{10.15}$$

Under this exchange of fields $d^2\lambda^a = J\, d^2\tilde{\lambda}^a$, and the Jacobian is

$$J = \left|\frac{\partial \tilde{\lambda}_m^a}{\partial \lambda_n^b}\right| = \begin{vmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & d & b \\ 0 & 0 & c & a \end{vmatrix} = (a\,d - b\,c)^2. \tag{10.16}$$

The $\mathrm{SL}(2, \mathbb{R})$ invariance now expresses itself completely as $\lambda^a \to \tilde{\lambda}^a$, and we rewrite the amplitude in terms of the original variable, ρ, and this new twistor field, $\tilde{\lambda}^a$. The progression of the invariance on the amplitude goes as

$$\int \prod_r \frac{{\pi_r}^1}{(\lambda^1(\rho_r))^2} \prod_r \delta\left({\pi_r}^2 - \frac{\lambda^2(\rho_r)}{\lambda^1(\rho_r)}{\pi_r}^1\right)(\rho_1-\rho_2)^2\left(\frac{{\pi_1}^1{\pi_2}^1}{\lambda^1(\rho_1)\lambda^1(\rho_2)}\right)^2$$
$$\times (\det\lambda)^3\, \delta^4(\Sigma_r \pi_r\bar{\pi}_r)\prod_a d^2\lambda^a \prod_r d\rho_r/d\gamma_M d\gamma_S\left[-\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}C_{0(3)}\right]$$
$$\longrightarrow \int \prod_r \frac{{\pi_r}^1}{(\lambda^1(\rho'_r))^2} \prod_r \delta\left({\pi_r}^2 - \frac{\lambda^2(\rho'_r)}{\lambda^1(\rho'_r)}{\pi_r}^1\right)(\rho'_1-\rho'_2)^2\left(\frac{{\pi_1}^1{\pi_2}^1}{\lambda^1(\rho'_1)\lambda^1(\rho'_2)}\right)^2$$
$$\times (\det\lambda)^3\, \delta^4(\Sigma_r \pi_r\bar{\pi}_r)\prod_a d^2\lambda^a \prod_r d\rho'_r/d\gamma_M d\gamma_S\left[-\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}C_{0(3)}\right]$$
$$= \int \prod_r \frac{{\pi_r}^1}{(\tilde{\lambda}^1(\rho_r))^2} \prod_r \delta\left({\pi_r}^2 - \frac{\tilde{\lambda}^2(\rho_r)}{\tilde{\lambda}^1(\rho_r)}{\pi_r}^1\right)(\rho_1-\rho_2)^2\left(\frac{{\pi_1}^1{\pi_2}^1}{\tilde{\lambda}^1(\rho_1)\tilde{\lambda}^1(\rho_2)}\right)^2$$
$$\times (\det\tilde{\lambda})^3\, \delta^4(\Sigma_r \pi_r\bar{\pi}_r)\prod_a d^2\tilde{\lambda}^a \prod_r d\rho_r/d\gamma_M d\gamma_S\left[-\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}C_{0(3)}\right]. \tag{10.17}$$

As we see, the amplitude is $\mathrm{SL}(2,\mathbb{R})$ invariant. And, the invariant measures can be expressed as $\prod_a d^2\lambda^a/(\det\lambda)^2 = d\gamma_M d\gamma_S$.

We continue the calculation of the amplitude (10.7) by making the change of variables $\zeta_r = \lambda^2(\rho_r)/\lambda^1(\rho_r)$. Under this change,

$$d\zeta_r = \frac{\det\lambda}{(\lambda^1(\rho_r))^2}d\rho_r, \tag{10.18}$$

and

$$\zeta_i - \zeta_j = \frac{\det\lambda}{\lambda^1(\rho_i)\lambda^1(\rho_j)}(\rho_i - \rho_j). \tag{10.19}$$

We make the change of variables to ζ's and integrate

$$\begin{aligned}
&\langle A_{-1}^{A_1}(\rho_1) A_{-1}^{A_2}(\rho_2) C(\rho_3)\rangle_{\text{tree}} \\
&\quad = \int \prod_r {\pi_r}^1 \prod_r \delta\left({\pi_r}^2 - \zeta_r {\pi_r}^1\right) \left(\frac{{\pi_1}^1 {\pi_2}^1 (\zeta_1 - \zeta_2)}{\det \lambda}\right)^2 \\
&\qquad \times \delta^4(\Sigma_r \pi_r \bar{\pi}_r) \prod_a d^2\lambda^a \prod_r d\zeta_r / d\gamma_M d\gamma_S \left[-\delta^{A_1 A_2} A_{-1(1)} A_{-1(2)} C_{0(3)}\right] \\
&\quad = -\delta^{A_1 A_2} \langle 12\rangle^2 \delta^4(\Sigma_r \pi_r \bar{\pi}_r) A_{-1(1)} A_{-1(2)} C_{0(3)}. \qquad (10.20)
\end{aligned}$$

The gluon polarizations are given by $\epsilon_r^- = A_{-1(r)} \pi_{ra} s_{r\dot{a}}$ and $\epsilon_r^+ = A_{1(r)} \bar{s}_{ra} \bar{\pi}_{r\dot{a}}$. We use momentum conservation and thus $s_{s\dot{b}} \sum_r \pi_r^b \bar{\pi}_r^{\dot{b}} = 0$ to find $s_{1\dot{b}} \bar{\pi}_2^{\dot{b}} = \frac{\langle 31\rangle}{\langle 23\rangle}$ and $s_{2\dot{b}} \bar{\pi}_1^{\dot{b}} = \frac{\langle 23\rangle}{\langle 31\rangle}$, so that $\epsilon_1^- \cdot p_2 \, \epsilon_2^- \cdot p_1 = -\langle 12\rangle^2 \, A_{-1(1)} A_{-1(2)}$, with $\langle rs\rangle = \pi_{ra} \pi_s^a$ and $[rs] = \bar{\pi}_{r\dot{a}} \bar{\pi}^{\dot{a}}$. We can set the scalar wave function $C_{0(3)} = 1$. Returning to the amplitude (10.2), we calculate the result,

$$\langle A_{-1}^{A_1}(\rho_1) A_{-1}^{A_2}(\rho_2) C(\rho_3)\rangle_{\text{tree}} = \epsilon_1^- \cdot p_2 \, \epsilon_2^- \cdot p_1 \, \delta^4(\Sigma \pi_r \bar{\pi}_r) \, \delta^{A_1 A_2} C_{0(3)}. \qquad (10.21)$$

We use the same method for calculating all subsequent $d = 1$ n-point trees, as we shall see in the following chapter. Subsequent unprimed three-point MHV amplitudes follow the same basic strategy, and will be shortened for brevity.

For two gluons and a graviton ($\phi\phi G$),

$$
\begin{aligned}
&\langle A_1^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)e_{-2}(\rho_3)\rangle_{\text{tree}} = \int \langle 0|e^{q_0}A_1^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)e_{-2}(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\
&= -\int \prod_{r=1}^{3} dk_r \prod_{r=1}^{3} k_r\ \lambda_a(\rho_3)\partial\lambda^a(\rho_3) \prod_{r,a} \delta({\pi_r}^a - k_r\lambda^a(\rho_r))\ (\rho_2-\rho_3)^4(k_2k_3)^4 \\
&\qquad \times \langle 0|e^{q_0}e^{\sum_{r=1}^{3} ik_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}|0\rangle \prod_a d^2\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M \left(\frac{\delta^{A_1A_2}A_{1(1)}A_{-1(2)}e_{-2(3)}}{(\rho_1-\rho_2)^2(k_1k_2)^2}\right) \\
&= -\delta^4(\Sigma\pi_r\bar{\pi}_r)\ \int \prod_{r=1}^{3} d\zeta_r \prod_{r=1}^{3} \pi_r^1\ \delta(\pi_r^2 - \zeta_r\pi_r^1)(\zeta_2-\zeta_3)^4(\pi_2^1\pi_3^1)^4(\zeta_1-\zeta_2)^{-2}(\pi_1^1\pi_2^1)^{-2} \\
&\qquad \times \delta^{A_1A_2}A_{1(1)}A_{-1(2)}e_{-2(3)} \\
&= -\delta^4(\Sigma\pi_r\bar{\pi}_r)\ \frac{\langle 23\rangle^4}{\langle 12\rangle^2}\ \delta^{A_1A_2}A_{1(1)}A_{-1(2)}e_{-2(3)} \\
&= \left(\epsilon_1^+\cdot\epsilon_2^-\ \epsilon^-_{3\,a\dot{a}b\dot{b}}p_1^{a\dot{a}}p_2^{b\dot{b}} + \epsilon_1^+\cdot p_2\ \epsilon^-_{3\,a\dot{a}b\dot{b}}\epsilon_2^{-a\dot{a}}p_2^{b\dot{b}} + \epsilon_2^-\cdot p_3\ \epsilon^-_{3\,a\dot{a}b\dot{b}}\epsilon_1^{+\,a\dot{a}}p_2^{b\dot{b}}\right)\delta^{A_1A_2}\ \delta^4(\Sigma\pi_r\bar{\pi}_r).
\end{aligned}
\tag{10.22}
$$

The gravity polarizations are $\epsilon_r^- = e_{-2(r)}\pi_{ra}s_{r\dot{a}}\pi_{rb}s_{r\dot{b}}$ and $\epsilon_r^+ = e_{2(r)}\bar{s}_{ra}\bar{\pi}_{r\dot{a}}\bar{s}_{rb}\bar{\pi}_{r\dot{b}}$, and one can factor

$$
\begin{aligned}
&\epsilon_1^+\cdot\epsilon_2^-\ \epsilon^-_{3\,a\dot{a}b\dot{b}}p_1^{a\dot{a}}p_2^{b\dot{b}} + \epsilon_1^+\cdot p_2\ \epsilon^-_{3\,a\dot{a}b\dot{b}}\epsilon_2^{-a\dot{a}}p_2^{b\dot{b}} + \epsilon_2^-\cdot p_3\ \epsilon^-_{3\,a\dot{a}b\dot{b}}\epsilon_1^{+\,a\dot{a}}p_2^{b\dot{b}} \\
&\qquad = \left(\epsilon_1^+\cdot\epsilon_2^-\ \epsilon_3^-\cdot p_1 + \epsilon_1^+\cdot p_2\ \epsilon_3^-\cdot\epsilon_2^- + \epsilon_2^-\cdot p_3\ \epsilon_3^-\cdot\epsilon_1^+\right)\epsilon_3^-\cdot p_2 \\
&\qquad = \frac{\langle 23\rangle^3}{\langle 12\rangle\langle 31\rangle}\ \frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle} = \frac{\langle 23\rangle^4}{\langle 12\rangle^2}.
\end{aligned}
\tag{10.23}
$$

For two gravitons and a scalar (GGG),

$$\begin{aligned}
&\langle e_{-2}(\rho_1)e_{-2}(\rho_2)C(\rho_3)\rangle_{\text{tree}} \\
&\qquad = \int \langle 0|e^{q_0} e_{-2}(\rho_1)e_{-2}(\rho_2)C(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\
&\qquad = \int \prod_{r=1}^{3} dk_r \prod_{r=1}^{3} k_r \ \lambda_a(\rho_1)\partial\lambda^a(\rho_1)\, \lambda_b(\rho_2)\partial\lambda^b(\rho_2)\, \lambda_c(\rho_3)\partial\lambda^c(\rho_3) \\
&\qquad\qquad \times \prod_{ra} \delta(\pi_r^a - k_r\lambda^a(\rho_r))\ (\rho_1-\rho_2)^4 k_1^4 k_2^4 \langle 0|e^{q_0} e^{\sum_{r=1}^{3} ik_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}|0\rangle \\
&\qquad\qquad \times \prod_a d^2\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M\ e_{-2(1)}e_{-2(2)}C_{0(3)} \\
&\qquad = \delta^4(\Sigma\pi_r\bar{\pi}_r)\ \int \prod_{r=1}^{3} d\zeta_r \prod_{r=1}^{3} \pi_r^1\, \delta(\pi_r^2 - \zeta_r\pi_r^1)\,(\zeta_1-\zeta_2)^2(\pi_1^1\pi_2^1)^2\ e_{-2(1)}e_{-2(2)}C_{0(3)} \\
&\qquad = \delta^4(\Sigma\pi_r\bar{\pi}_r)\,\langle 12\rangle^4 e_{-2(1)}e_{-2(2)}C_{0(3)} \\
&\qquad = \epsilon^-_{1\,a\dot{a}b\dot{b}}\ p_2^{a\dot{a}}p_2^{b\dot{b}}\ \epsilon^-_{2\,c\dot{c}d\dot{d}}p_1^{c\dot{c}}p_1^{d\dot{d}}\ \ \delta^4(\Sigma\pi_r\bar{\pi}_r)\ \ C_{0(3)}.
\end{aligned} \tag{10.24}$$

These are conventional couplings.

Less conventional is the three-graviton coupling (GGF):

$$\begin{aligned}
&\langle e_{-2}(\rho_1)e_{-2}(\rho_2)e_2(\rho_3)\rangle_{\text{tree}} = \int \langle 0|e^{q_0} e_{-2}(\rho_1)e_{-2}(\rho_2)e_2(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\
&= i\int \prod_{r=1}^{3} dk_r \frac{k_1k_2}{k_3^2}\ \lambda_a(\rho_1)\partial\lambda^a(\rho_1)\,\lambda_a(\rho_2)\partial\lambda^a(\rho_2)\ \prod_{ra}\delta(\pi_r^a - k_r\lambda^a(\rho_r))\ (\rho_1-\rho_2)^4 k_1^4k_2^4 \\
&\qquad \times \langle 0|e^{q_0} e^{\sum_{r=1}^{3} ik_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}\ \bar{\pi}_3^{\dot{a}}Y_{\dot{a}}(\rho_3)|0\rangle \prod_a d^2\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M e_{-2(1)}e_{-2(2)}e_{2(3)} \\
&= \delta^4(\Sigma\pi_r\bar{\pi}_r)\int \prod_{r=1}^{3} d\zeta_r \prod_{r=1}^{3}\delta(\pi_r^2-\zeta_r\pi_r^1)(\zeta_1-\zeta_2)^4(\pi_1^1\pi_2^1)^4\frac{\pi_1^1\pi_2^1}{(\pi_3^1)^2}\sum_{r=1}^{2}\frac{\pi_r^1[3r]}{\zeta_r-\zeta_3}e_{-2(1)}e_{-2(2)}e_{2(3)} \\
&= \delta^4(\Sigma\pi_r\bar{\pi}_r)\langle 12\rangle^4\ \ \sum_{r=1}^{2}\frac{[3r]\langle r\xi\rangle^2}{\langle 3r\rangle\langle 3\xi\rangle^2}\, e_{-2(1)}e_{-2(2)}e_{2(3)} \\
&= \delta^4(\Sigma\pi_r\bar{\pi}_r)\,\langle 12\rangle^6\ \frac{[32]}{\langle 32\rangle\langle 31\rangle^2}\, e_{-2(1)}e_{-2(2)}e_{2(3)} = 0.
\end{aligned} \tag{10.25}$$

We calculate the last step due to momentum conservation, $\sum_r p_{r\,a\dot{a}} = 0$,

$$\begin{aligned}\sum p_{r\,a\dot{a}} &= \pi_{1a}\bar{\pi}_{1\dot{a}} + \pi_{2a}\bar{\pi}_{2\dot{a}} + \pi_{3a}\bar{\pi}_{3\dot{a}} \qquad \text{(multiply by } {\pi_1}^a\text{)} \\ &= \langle 12\rangle\,\bar{\pi}_{2\dot{a}} + \langle 13\rangle\,\bar{\pi}_{3\dot{a}} \qquad\qquad \text{(multiply by } {\bar{\pi}_3}^{\dot{a}}\text{)} \\ &= \langle 12\rangle[23] = 0. \end{aligned} \tag{10.26}$$

Since this amplitude involves $Y_{\dot{a}}$, we have first evaluated, using (9.22),

$$\langle 0|e^{q_0} e^{\sum_{r=1}^3 ik_r \bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}\,\bar{\pi}_3^{\dot{a}} Y_{\dot{a}}(\rho_3)|0\rangle = -i\sum_{r\neq 3}\frac{k_r[3r]}{(\rho_r-\rho_3)}\langle 0|e^{q_0} e^{\sum_{r=1}^3 ik_r \bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}|0\rangle, \tag{10.27}$$

then replaced $\mu^{\dot{b}}(\rho)$ by its lowest modes and changed variables from ρ_r to ζ_r, as discussed in more detail in (11.8). The expression is independent of the spinor ξ. We compare this vanishing three-graviton tree amplitude for conformal gravity with that of Einstein gravity,

$$\begin{aligned}\langle e_{-2}(\rho_1)e_{-2}(\rho_2)e_2(\rho_3)\rangle_{Einstein\,tree} &= \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}\,\delta^4(\Sigma\pi_r\bar{\pi}_r)\,e_{-2(1)}e_{-2(2)}e_{2(3)} \neq 0 \\ &= \frac{1}{s_{23}}\langle e_{-2}(\rho_1)e_{-2}(\rho_2)e_2(\rho_3)\rangle_{\text{tree}}. \end{aligned} \tag{10.28}$$

For two scalars and a graviton (GGF), the amplitude also vanishes by momentum conservation:

$$
\begin{aligned}
\langle C(\rho_1)e_{-2}(\rho_2)\bar{C}(\rho_3)\rangle_{\rm tree} &= \int \langle 0|e^{q_0}C(\rho_1)e_{-2}(\rho_2)\bar{C}(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\
&= i\int \prod_{r=1}^{3} dk_r \frac{k_1k_2}{k_3^2}\ \lambda_a(\rho_1)\partial\lambda^a(\rho_1)\ \lambda_a(\rho_2)\partial\lambda^a(\rho_2)\ \prod_{ra}\delta(\pi_r^a - k_r\lambda^a(\rho_r))\ (\rho_2-\rho_3)^4 k_2^4 k_3^4 \\
&\qquad \times \langle 0|e^{q_0}e^{\sum_{r=1}^{3} ik_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}\ \bar{\pi}_3^{\dot{a}}Y_{\dot{a}}(\rho_3)|0\rangle \prod_a d^2\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M\ C_{0(1)}e_{-2(2)}\bar{C}_{0(3)} \\
&= \delta^4(\Sigma\pi_r\bar{\pi}_r)\ \langle 12\rangle^2\ \frac{[23]\langle 23\rangle^3}{\langle 31\rangle^2} C_{0(1)}e_{-2(2)}\bar{C}_{0(3)} = 0.
\end{aligned}
\tag{10.29}
$$

The familiar degree one three-point gluon vertex,

$$
\begin{aligned}
&\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)A_{1}^{A_3}(\rho_3)\rangle_{\rm tree} \\
&\quad = \int \langle 0|e^{q_0}A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)A_{1}^{A_3}(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\
&\quad = \int \prod_{r=1}^{3} \frac{dk_r}{k_r} \prod_{ra}\delta(\pi_r^a - k_r\lambda^a(\rho_r))\ (\rho_1-\rho_2)^4 k_1^4 k_2^4\ \frac{f^{A_1A_2A_3}}{(\rho_1-\rho_2)(\rho_2-\rho_3)(\rho_3-\rho_1)} \\
&\qquad\qquad \times \langle 0|e^{q_0}e^{\sum_{r=1}^{3} ik_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}|0\rangle \prod_a d^2\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M\ A_{-1(1)}A_{-1(2)}A_{1(3)} \\
&\quad = \delta^4(\Sigma\pi_r\bar{\pi}_r)\ \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\ f^{A_1A_2A_3}\ A_{-1(1)}A_{-1(2)}A_{1(3)} \\
&\quad = \delta^4(\Sigma\pi_r\bar{\pi}_r)\, f^{A_1A_2A_3}\ \left(\epsilon_1^-\cdot\epsilon_2^-\epsilon_3^+\cdot p_1 + \epsilon_2^-\cdot\epsilon_3^+\epsilon_1^-\cdot p_2 + \epsilon_3^+\cdot\epsilon_1^-\epsilon_2^-\cdot p_3\right).
\end{aligned}
\tag{10.30}
$$

The remaining unprimed three-point functions with two negative helicity states vanish. The unprimed MHV three-point functions are summarized in Table 10.1, where we include their polarizations and momentum conserving delta function, in order to compare with primed couplings in Table 10.3.

We compare these couplings with those of opposite helicities, with instanton number $d = 0$. A slightly different calculational strategy is involved, most especially the Mobius

$\langle A_{-1}^{A_1} A_{-1}^{A_2} C\rangle = -\langle 12\rangle^2\, \delta^{A_1 A_2} A_{-1(1)} A_{-1(2)} C_{0(3)} \delta^4(\Sigma\pi_r\bar{\pi}_r)$
$\langle A_1^{A_1} A_{-1}^{A_2} e_{-2}\rangle = -\frac{\langle 23\rangle^4}{\langle 12\rangle^2} \delta^{A_1 A_2} A_{1(1)} A_{-1(2)} e_{-2(3)} \delta^4(\Sigma\pi_r\bar{\pi}_r)$
$\langle e_{-2} e_{-2} C\rangle = \langle 12\rangle^4 e_{-2(1)} e_{-2(2)} C_{0(3)} \delta^4(\Sigma\pi_r\bar{\pi}_r)$
$\langle e_{-2} e_{-2} e_2\rangle = \frac{\langle 12\rangle^6 [23]}{\langle 23\rangle\langle 31\rangle^2} e_{-2(1)} e_{-2(2)} e_{2(3)} \delta^4(\Sigma\pi_r\bar{\pi}_r) = 0$
$\langle C e_{-2} \bar{C}\rangle = \frac{\langle 12\rangle^2\langle 23\rangle^3 [23]}{\langle 31\rangle^2} C_{0(1)} e_{-2(2)} \bar{C}_{0(3)} \delta^4(\Sigma\pi_r\bar{\pi}_r) = 0$
$\langle A_{-1}^{A_1} A_{-1}^{A_2} A_1^{A_3}\rangle = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} f^{A_1 A_2 A_3} A_{-1(1)} A_{-1(2)} A_{1(3)} \delta^4(\Sigma\pi_r\bar{\pi}_r)$

Table 10.1: Unprimed conformal supergravity MHV couplings

invariance of $\mathrm{SL}(2,\mathbb{R})$. We show this more explicitly in the first amplitude.

$$
\begin{aligned}
\langle A_1^{A_1}(\rho_1) A_1^{A_2}(\rho_2) \bar{C}(\rho_3)\rangle_{\text{tree}} &= \int \langle 0| A_1^{A_1}(\rho_1) A_1^{A_2}(\rho_2) \bar{C}(\rho_3) |0\rangle \prod_{r=1}^{3} d\rho_r / d\gamma_M d\gamma_S \\
&= i \int \prod_{r=1}^{3} dk_r \frac{k_3^2}{k_1 k_2} \prod_{r,a} \delta({\pi_r}^a - k_r \lambda^a) \langle 0| e^{\sum_{r=1}^3 i k_r \bar{\pi}_{r\dot{b}} \mu^{\dot{b}}(\rho_r)}\ \bar{\pi}_3^{\dot{a}} Y_{\dot{a}}(\rho_3)\ |0\rangle \\
&\qquad \times \prod_a d\lambda^a \prod_r d\rho_r / d\gamma_S d\gamma_M \left(- \frac{\delta^{A_1 A_2} A_{1(1)} A_{1(2)} \bar{C}_{0(3)}}{(\rho_1 - \rho_2)^2} \right) \\
&= - \int \prod_{r=1}^{3} dk_r \frac{k_3^2}{k_1 k_2} \prod_{ra} \delta(\pi_r^a - k_r \lambda^a) \left(\sum_{r \neq 3} \frac{k_r [3r]}{(\rho_r - \rho_3)} \right) \prod_{a=1}^{2} \delta(\sum_{r=1}^{3} k_r \bar{\pi}_{ra}) \\
&\qquad \times \prod_a d\lambda^a \prod_r d\rho_r / d\gamma_S d\gamma_M\ \delta^{a_1 a_2} A_{1(1)} A_{1(2)} \bar{C}_{0(3)}\ (\rho_1 - \rho_2)^{-2} \\
&= -\delta^{a_1 a_2} [31] \frac{(\pi_3^1)^2}{\pi_2^1} \int \frac{d\lambda^2}{\lambda^1} \prod_{r=1}^{3} \delta(\pi_r^2 - \frac{\lambda^2}{\lambda^1} \pi_r^1) \prod_a \delta \left(\sum_{r=1}^{3} \frac{\pi^1}{\lambda^1} \bar{\pi}_{ra} \right)\ A_{1(1)} A_{1(2)} \bar{C}_{0(3)} \\
&= -\, \delta^{A_1 A_2}\ \delta^4(\Sigma \pi_r \bar{\pi}_r)\ [12]^2\, A_{1(1)} A_{1(2)} \bar{C}_{0(3)}.
\end{aligned} \tag{10.31}
$$

After eliminating $Y_{\dot{b}}$, for degree $d = 0$, we replace $Z^I(\rho)$ with Z_0^I. We have $\pi_2^1[21] = -\pi_3^1[31]$, replacing the product $\prod_{r=2}^{3} \delta(\pi_r^2 - \frac{\lambda^2}{\lambda^1}\pi_r^1)$ with $\prod_{a=1}^{2} \delta(\sum_{r=1}^{3}(\pi_r^2 - \frac{\lambda^2}{\lambda^1}\pi_r^1)\bar{\pi}_{ra})[23] = \prod_{a=1}^{2} \delta(\sum_{r=1}^{3} \pi_r^2 \bar{\pi}_{ra})[23]$. Here $\langle 0|\psi_0^1|0\rangle = 1$, see reference [51]. This amplitude is type

$\phi\phi F$. It is useful to express the invariant measures as

$$d\gamma_M = \prod_{r=1}^{3} d\rho_r \frac{1}{(\rho_1-\rho_2)(\rho_2-\rho_3)(\rho_3-\rho_1)} \tag{10.32}$$

and

$$d\gamma_S = \frac{d\lambda^1}{\lambda^1} \tag{10.33}$$

in the $d=0$ sector, where $\lambda^a = \lambda^a_0$.

If we compare the amplitude in (10.31) with (10.21), we verify that the $d = 0$ tree $\langle A_1^{A_1}(\rho_1)A_1^{A_2}(\rho_2)\bar{C}(\rho_3)\rangle_{\text{tree}}$ is the antiholomorphic version of the $d=1$ coupling, $\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C(\rho_3)\rangle_{\text{tree}}$. Similarly, the $\phi\phi F$ tree

$$\begin{aligned}
\langle A_{-1}^{A_1}(\rho_1)A_1^{A_2}(\rho_2)e_2(\rho_3)\rangle_{\text{tree}} &= \int \langle 0|A_{-1}^{A_1}(\rho_1)A_1^{A_2}(\rho_2)e_2(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\
&= i\int \prod_{r=1}^{3} dk_r \, \frac{k_1^3}{k_2k_3^2} \prod_{ra} \delta(\pi_r^a - k_r\lambda^a)\langle 0|e^{\sum_{r=1}^{3} ik_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}\bar{\pi}_3^{\dot{a}}Y_{\dot{a}}(\rho_3)|0\rangle \\
&\qquad \times \prod_a d\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M \left(\frac{-\delta^{A_1A_2}A_{-1(1)}A_{1(2)}e_{2(3)}}{(\rho_1-\rho_2)^2}\right) \\
&= -\delta^{A_1A_2}[31]\frac{(\pi_3{}^1)^2}{\pi_2{}^1}\int \frac{d\lambda^2}{\lambda^1}\prod_{r=1}^{3}\delta(\pi_r{}^2 - \frac{\lambda^2}{\lambda^1}\pi_r{}^1)\prod_a \delta(\sum_{r=1}^{3}\frac{\pi_r{}^1}{\lambda^1}\bar{\pi}_{ra})A_{-1(1)}A_{1(2)}e_{2(3)} \\
&= -\delta^{A_1A_2}\ \delta^4(\Sigma\pi_r\bar{\pi}_r)\ \frac{[23]^4}{[12]^2}\,A_{-1(1)}A_{1(2)}e_{2(3)},
\end{aligned} \tag{10.34}$$

is the antiholomorphic version of (10.22). The FFF amplitude

$$\begin{aligned}
\langle e_2(\rho_1)e_2(\rho_2)\bar{C}(\rho_3)\rangle_{\text{tree}} &= \int \langle 0|e_2(\rho_1)e_2(\rho_2)\bar{C}(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\
&= -i\int \prod_{r=1}^{3} dk_r\, \frac{k_3^2}{k_1^2k_2^2} \prod_{ra} \delta({\pi_r}^a - k_r\lambda^a)\ \langle 0| \prod_{r=1}^{3}\ e^{ik_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}\ \bar{\pi}_r^{\dot{a}}Y_{\dot{a}}(\rho_r)|0\rangle \\
&\qquad \times \prod_a d\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M\ e_{2(1)}e_{2(2)}\bar{C}_{0(3)} \\
&= -i\int \prod_{r=1}^{3} dk_r\, \frac{k_3^2}{k_1^2k_2^2} \prod_{ra} \delta(\pi_r^a - k_r\lambda^a(\rho_r))\ (\frac{-i}{\lambda^1})^3\ {\pi_2}^1({\pi_1}^1)^2 \\
&\qquad \times \frac{[12]^2[31]}{(\rho_1-\rho_2)(\rho_2-\rho_3)(\rho_3-\rho_1)} \prod_{\dot{a}} \delta(\Sigma_{r=1}^3 k_r\bar{\pi}_{r\dot{a}}) \\
&\qquad \times \prod_a d\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M\ e_{2(1)}e_{2(2)}\bar{C}_{0(3)} \\
&= \delta^4(\Sigma\pi_r\bar{\pi}_r)\ [12]^4\ e_{2(1)}e_{2(2)}\bar{C}_{0(3)}
\end{aligned} \tag{10.35}$$

is the antiholomorphic version of (10.24).

Of course, we expect these results for the $d = 0$ amplitudes, from the parity properties of the vertex operators. But we present the derivations to demonstrate our computational methods, and to verify (10.25). The $d = 0$ three-graviton coupling vanishes identically, since the vertex operator $e_{-2}(\rho)$ involves $\lambda_a(\rho)\partial\lambda^a(\rho)$, which vanishes for $\lambda^a(\rho) = \lambda_0^a$, a constant:

$$\begin{aligned}
\langle e_2(\rho_1)e_2(\rho_2)e_{-2}(\rho_3)\rangle_{\text{tree}} &= \int \langle 0|e_2(\rho_1)e_2(\rho_2)e_{-2}(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\
= -\int \prod_{r=1}^{3} dk_r\, \frac{k_3^5}{k_1^2k_2^2} \prod_{ra} &\delta(\pi_r^a - k_r\lambda^a(\rho_r))\ \langle 0| \prod_{r=1}^{2} e^{ik_r\bar{\pi}_{rb}\mu^b(\rho_r)}\ \bar{\pi}_r^{\dot{a}}Y_{\dot{a}}(\rho_r)\ e^{ik_3\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_3)}|0\rangle \\
&\times \lambda_a(\rho_3)\partial\lambda^a(\rho_3)\ \prod_a d\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M\ e_{2(1)}e_{2(2)}e_{-2(3)} = 0,
\end{aligned} \tag{10.36}$$

and

$$\langle \bar{C}(\rho_1)e_2(\rho_2)C(\rho_3)\rangle_{\text{tree}} = \int \langle 0|\bar{C}(\rho_1)e_2(\rho_2)C(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S$$
$$= -\int \prod_{r=1}^{3} dk_r \, \frac{k_1^2 k_3}{k_2^2} \prod_{ra} \delta(\pi_r^a - k_r\lambda^a) \, \langle 0| \prod_{r=1}^{2} e^{ik_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)} \, \bar{\pi}_r^{\dot{a}} Y_{\dot{a}}(\rho_r) \, e^{ik_3\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_3)}|0\rangle$$
$$\times \lambda_a(\rho_3)\partial\lambda^a(\rho_3) \, \prod_a d\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M \, \bar{C}_{0(1)} e_{2(2)} C_{0(3)} = 0. \tag{10.37}$$

These are FFG trees. Finally, we include the familiar degree zero three-gluon vertex

$$\langle A_1^{A_1}(\rho_1)A_1^{A_2}(\rho_2)A_{-1}^{A_3}(\rho_3)\rangle_{\text{tree}}$$
$$= \int \langle 0|A_1^{A_1}(\rho_1)A_1^{A_2}(\rho_2)A_{-1}^{A_3}(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S$$
$$= \int \prod_{r=1}^{3} dk_r \frac{k_3^3}{k_1k_2} \prod_{ra} \delta(\pi_r^a - k_r\lambda^a) \, \frac{f^{A_1A_2A_3}}{(\rho_1-\rho_2)(\rho_2-\rho_3)(\rho_3-\rho_1)}$$
$$\times \langle 0|e^{\sum_{r=1}^{3} ik_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}|0\rangle \prod_a d\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M \, A_{1(1)}A_{1(2)}A_{-1(3)}$$
$$= \delta^4(\Sigma\pi_r\bar{\pi}_r) \, f^{A_1A_2A_3} \, \frac{[12]^3}{[23][31]} \, A_{1(1)}A_{1(2)}A_{-1(3)}. \tag{10.38}$$

We express the $d = 0$ three-point amplitudes in the table below.

$\langle A_1^{A_1}A_1^{A_2}\bar{C}\rangle = -[12]^2\delta^{A_1A_2}A_{1(1)}A_{1(2)}\bar{C}_{0(3)}\delta^4(\Sigma\pi_r\bar{\pi}_r)$
$\langle A_{-1}^{A_1}A_1^{A_2}e_2\rangle = -\frac{[23]^4}{[12]^2}\delta^{A_1A_2}A_{-1(1)}A_{1(2)}e_{2(3)}\delta^4(\Sigma\pi_r\bar{\pi}_r)$
$\langle e_2e_2\bar{C}\rangle = [12]^4 e_{2(1)}e_{2(2)}\bar{C}_{0(3)}\delta^4(\Sigma\pi_r\bar{\pi}_r)$
$\langle e_2e_2e_{-2}\rangle = 0$
$\langle \bar{C}e_2C\rangle = 0$
$\langle A_1^{A_1}A_1^{A_2}A_{-1}^{A_3}\rangle = \frac{[12]^3}{[23][31]}f^{A_1A_2A_3}A_{1(1)}A_{1(2)}A_{-1(3)}\delta^4(\Sigma\pi_r\bar{\pi}_r)$

Table 10.2: $d = 0$ Unprimed conformal supergravity couplings

10.2 Amplitudes with Primed Vertices

In this section we will compute tree amplitudes containing states with primed vertex operators. As a preliminary study, consider the $d = 1$ coupling $\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C'(\rho_3)\rangle_{\text{tree}}$ with $\phi\phi G'$ vertex operators. Using the previous methods, it is convenient to evaluate the primed coupling as

$$
\begin{aligned}
&\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C'(\rho_3)\rangle_{\text{tree}} = \int \langle 0|e^{q_0} A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C'(\rho_3)|0\rangle \\
&= -\int \prod_{r=1}^{3} dk_r \prod_{r=1}^{3} k_r \prod_{ra} \delta({\pi_r}^a - k_r\lambda^a(\rho_r))\, k_1^2 k_2^2\, (\rho_1-\rho_2)^2 \prod_{r=1}^{3} d\rho_r \\
&\quad \times \langle 0|e^{q_0}\, i \left(\frac{s_{3\dot a}}{k_3}\partial\mu^{\dot a}(\rho_3) - s_{3\dot a}\mu^{\dot a}(\rho_3)\bar s_{3a}\partial\lambda^a(\rho_3)\right)\; e^{i\sum_{r=1}^{3} k_r \bar\pi_{r\dot b}\mu^{\dot b}(\rho_r)}|0\rangle \\
&\quad \times \prod_a d^2\lambda^a/d\gamma_M d\gamma_S \delta^{A_1A_2} A_{-1(1)}A_{-1(2)}C'_{0(3)} \\
&= -\int \prod_{r=1}^{3} \frac{\pi_r^1}{(\lambda^1(\rho_r))^2}\, \delta\left(\pi_r^2 - \frac{\lambda^2(\rho_r)}{\lambda^1(\rho_r)}\pi_r^1\right) (\rho_1-\rho_2)^2 \left(\frac{\pi_1^1\pi_2^1}{\lambda^1(\rho_1)\lambda^1(\rho_2)}\right)^2 \\
&\quad \times \prod_{a,\dot a} d^2\lambda^a d^2\mu^{\dot a} \prod_r d\rho_r/d\gamma_S d\gamma_M\, \delta^{A_1A_2}\; A_{-1(1)}A_{-1(2)}C'_{0(3)} \\
&\quad \times i\Big(\frac{s_{31}}{\pi_3^1}\; (\lambda_0^1\mu_{-1}^1 - \mu_0^1\lambda_{-1}^1) + \frac{s_{32}}{\pi_3^2}\; (\lambda_0^2\mu_{-1}^2 - \mu_0^2\lambda_{-1}^2)\Big) e^{i\sum_{r=1}^{3}\frac{\pi_r^1\bar\pi_{r\dot b}}{\lambda^1(\rho_r)}(\mu_0^{\dot b}+\rho_r\mu_{-1}^{\dot b})}
\end{aligned}
\tag{10.39}
$$

where we have used the delta functions $\delta(\pi_r^1 - k_r\lambda^1(\rho_r))$ to do the k_r integrations. Here $\lambda^a(\rho) = \lambda_0^a + \rho\lambda_{-1}^a$. In order to perform the $d^2\mu^{\dot a}$ integrations, we note that

$$
\sum_{r=1}^{n} \pi_r^b\bar\pi_{r\dot a} = \sum_{r=1}^{n} \frac{\lambda^b(\rho_r){\pi_r}^1\bar\pi_{r\dot a}}{\lambda^1(\rho_r)} = \lambda_0^b \sum_{r=1}^{n} \frac{\pi_r^1\bar\pi_{r\dot a}}{\lambda^1(\rho_r)} + \lambda_{-1}^b \sum_{r=1}^{n} \frac{\pi_r^1\bar\pi_{r\dot a}\rho_r}{\lambda^1(\rho_r)} \tag{10.40}
$$

for any n, when $\pi_r^2 - (\lambda^2(z_r)/\lambda^1(z_r))\pi_r^1 = 0$. We can invert this change of variables to write $\sum_{r=1}^{2} \frac{\pi_r^1}{\lambda^1(\rho_r)}\bar\pi_{r\dot a}$ and $\sum_{r=1}^{2} \frac{\pi_r^1}{\lambda^1(\rho_r)}\bar\pi_{r\dot a}\rho_r$ in terms of $\sum_{r=1}^{2}\pi^b\bar\pi_{\dot a}$, and express the exponential in (10.39) as

$$e^{i\sum_{r=1}^{3}\frac{\pi_r^1}{\lambda^1(\rho_r)}\bar{\pi}_{r\dot{b}}(\mu_0^{\dot{b}}+\rho_r\mu_{-1}^{\dot{b}})} = e^{i\frac{\epsilon_{ca}}{\det\lambda}\left(\lambda_0^a\mu_{-1}^{\dot{b}}-\lambda_{-1}^a\mu_0^{\dot{b}}\right)\sum_{r=1}^{3}\pi_r^c\bar{\pi}_{r\dot{b}}}, \tag{10.41}$$

where the anti-symmetric epsilon tensor is $\epsilon^{12} = 1 = -\epsilon_{12}$, as in Chapter 9. Then the integrand of the $d^2\mu^{\dot{a}}$ integrations can be expressed as derivatives of the exponential,

$$\begin{aligned}
&\int\prod_{\dot{a}} d^2\mu^{\dot{a}}\left(\frac{s_{31}}{\pi_3^1}\left(\lambda_0^1\mu_{-1}^1-\mu_0^1\lambda_{-1}^1\right)+\frac{s_{32}}{\pi_3^2}\left(\lambda_0^2\mu_{-1}^2-\mu_0^2\lambda_{-1}^2\right)\right)e^{i\sum_{r=1}^{3}\frac{\pi_r^1}{\lambda^1(\rho_r)}\bar{\pi}_{r\dot{b}}(\mu_0^{\dot{b}}+\rho_r\mu_{-1}^{\dot{b}})}\\
&\qquad = -i\,(\det\lambda)\left(\frac{s_{31}}{\pi_3^1}\,\frac{\partial}{\partial\sum_{r=1}^{3}\pi_r^2\bar{\pi}_{r1}}-\frac{s_{32}}{\pi_3^2}\,\frac{\partial}{\partial\sum_{r=1}^{3}\pi_r^1\bar{\pi}_{r2}}\right)\\
&\qquad\qquad\times\int\prod_{\dot{a}} d^2\mu^{\dot{a}}e^{i\frac{\epsilon_{ca}}{\det\lambda}\left(\lambda_0^a\mu_{-1}^{\dot{b}}-\lambda_{-1}^a\mu_0^{\dot{b}}\right)\sum_{r=1}^{3}\pi_r^c\bar{\pi}_{r\dot{b}}}.
\end{aligned} \tag{10.42}$$

Performing the $d^2\mu^{\dot{a}}$ integrals to find momentum delta functions,

$$\int\prod_{\dot{a}} d^2\mu^{\dot{a}}e^{i\frac{\epsilon_{ca}}{\det\lambda}\left(\lambda_0^a\mu_{-1}^{\dot{b}}-\lambda_{-1}^a\mu_0^{\dot{b}}\right)\sum_{r=1}^{3}\pi_r^c\bar{\pi}_{r\dot{b}}} = (\det\lambda)^2\,\delta^4(\Sigma\pi_r\bar{\pi}_r), \tag{10.43}$$

and using our previous methods, (10.39) becomes

$$\begin{aligned}
&\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C'(\rho_3)\rangle_{\text{tree}}\\
&\quad = -\langle 12\rangle^2\delta^{A_1A_2}\Big[\frac{s_{31}}{\pi_3^1}\,\delta^2(\sum_{r=1}^{3}\pi_r^a\bar{\pi}_{r2})\,\delta(\sum_{r=1}^{3}\pi_r^1\bar{\pi}_{r1})\,\delta'\left(\sum_{r=1}^{3}\pi_r^2\bar{\pi}_{r1}\right)\\
&\qquad -\frac{s_{32}}{\pi_3^2}\,\delta^2(\sum_{r=1}^{3}\pi_r^a\bar{\pi}_{r1})\;\delta(\sum_{r=1}^{3}\pi_r^2\bar{\pi}_{r2})\,\delta'\left(\sum_{r=1}^{3}\pi_r^1\bar{\pi}_{r2}\right)\Big]\;A_{-1(1)}A_{-1(2)}C'_{0(3)}\\
&\quad = \langle 12\rangle^2\,\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}\frac{C'_{0(3)}}{2p_3^0}\frac{\partial}{\partial P^0}\delta^4(\Sigma\pi_r\bar{\pi}_r)
\end{aligned} \tag{10.44}$$

where we have chosen the Berkovits Witten gauge $s_{\dot{a}} = \frac{\pi^a\sigma^0_{a\dot{a}}}{2p^0}$, so $\frac{s_{21}}{\pi_2^1} = \frac{s_{22}}{\pi_2^2} = \frac{1}{\pi_2^1\bar{\pi}_2^1+\pi_2^2\bar{\pi}_2^2} = \frac{1}{2p_2^0}$, and defined $P^0 = \sum_{r=1}^{3}p_r^0 = \frac{1}{2}\sum_{r=1}^{3}(\pi_r^1\bar{\pi}_{r2}-\pi_r^2\bar{\pi}_{r1})$, using $p_{ra\dot{a}} = \pi_{ra}\bar{\pi}_{r\dot{a}} = \sigma^{\mu}_{a\dot{a}}p_{r\mu}$

as in Chapter 9.

We interpret the amplitude (10.44) with the help of understanding how the momentum operator acts on the primed states. In conformal supergravity, the dipole pairs arise as solutions to equations of motion with higher than quadratic derivatives, see for example [57, 35]. Each pair σ_p, σ'_p satisfies $(\partial_\mu \partial^\mu)^2 \sigma = 0$, and comprises a plane wave state $\sigma_p = e^{ip\cdot x}$, and a state $\sigma'_p = iA \cdot x e^{ip\cdot x}$ that cannot diagonalize the momentum operator for any non-zero vector A independent of x. Since $P^{\mathrm{op}}_{a\dot{a}} = -i\frac{\partial}{\partial x^{a\dot{a}}}$, then

$$P^{\mathrm{op}}_{a\dot{a}}\sigma_p = p_{a\dot{a}}\sigma_p, \qquad P^{\mathrm{op}}_{a\dot{a}}\sigma'_p = p_{a\dot{a}}\sigma'_p + A_{a\dot{a}}\sigma_p. \tag{10.45}$$

In particular, we can write $\sigma'_p = A^{a\dot{a}}\frac{\partial}{\partial p^{a\dot{a}}}\sigma_p$, and choose A to be in the time direction [35] to make contact with the Berkovits Witten gauge, so

$$\sigma'_p \sim \frac{\partial}{\partial p^0}\sigma_p. \tag{10.46}$$

The primed amplitude (10.44) is effectively $-\frac{C'_{0(3)}}{2p_3^0 C_{0(3)}}\frac{\partial}{\partial p_3^0}$ times the form (10.21), as expected in view of (10.46). For the pair of states to have the same relative dimension, the wavefunctions $C_{0(r)}, C'_{0(r)}$ differ in dimension by a factor of $(p_r^0)^2$, so the primed amplitude (10.44) has canonical dimensions.

But what about momentum conservation? Surely primed amplitudes are conformally invariant, just as the others. Although the primed states are not eigenstates of the momentum operator, we know they transform as in (10.45). So, the momentum operator acts on the coupling $\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C'(\rho_3)\rangle_{\text{tree}}$ as

$$
\begin{aligned}
P^0\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C'(\rho_3)\rangle_{\text{tree}} &- \frac{C'_{0(3)}}{C_{0(3)}2p_3^0}\,\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C(\rho_3)\rangle_{\text{tree}}\\
&= \langle 12\rangle^2\,\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}\frac{C'_{0(3)}}{2p_3^0}\;P^0\frac{\partial}{\partial P^0}\delta(P^0)\delta^3(P^i)\\
&\quad + \frac{C'_{0(3)}}{2p_3^0}\,\langle 12\rangle^2\,\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}\delta(P^0)\delta^3(P^i) = 0
\end{aligned}
\tag{10.47}
$$

and

$$
P^i\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C'(\rho_3)\rangle_{\text{tree}} = \langle 12\rangle^2\,\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}\;C'_{0(3)}\frac{\partial}{\partial P^0}\delta(P^0)P^i\delta^3(P^i) = 0,
\tag{10.48}
$$

verifying that the primed amplitude (10.44) has translational invariance. Here $P^\mu = \sum_r p_r^\mu$, and $P^0\frac{\partial}{\partial P^0}\delta(P^0) = -\delta(P^0)$ on the support of a test function.

In a similar calculation, now using the $e'_{-2}(\rho)$ vertex operator in lieu of $C'(\rho)$, we find the MHV coupling for two gluons and a primed graviton:

$$
\begin{aligned}
&\langle A_1^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)e'_{-2}(\rho_3)\rangle_{\text{tree}}\\
&\quad= \int \langle 0|e^{q_0}A_1^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)e'_{-2}(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S\\
&\quad= -\frac{\langle 23\rangle^4}{\langle 12\rangle^2}\delta^{A_1A_2}A_{1(1)}A_{-1(2)}e'_{-2(3)}\\
&\qquad\times\left[\frac{s_{31}}{\pi_3^1}\,\frac{\partial}{\partial\sum_{r=1}^3\pi_r^2\bar\pi_{r1}} - \frac{s_{32}}{\pi_3^2}\,\frac{\partial}{\partial\sum_{r=1}^3\pi_r^1\bar\pi_{r2}}\right]\delta^4\Big(\sum_{r=1}^{2}\pi_r\bar\pi_r\Big)\\
&\quad= \frac{\langle 23\rangle^4}{\langle 12\rangle^2}\,\delta^{A_1A_2}A_{1(1)}A_{-1(2)}\frac{e'_{-2(3)}}{2p_3^0}\frac{\partial}{\partial P^0}\delta^4(\Sigma\pi_r\bar\pi_r).
\end{aligned}
\tag{10.49}
$$

We can extend the $d=1$ GGG coupling of two gravitons and a scalar to any combination of primed vertices $V_{G'}$ as follows. If there is more than one primed vertex operator, there will be a product of factors in the derivation of the amplitude, of the form,

$$\Big(\frac{s_{r1}}{\pi_r^1}\ \left(\lambda_0^1\mu_{-1}^1 - \mu_0^1\lambda_{-1}^1\right) + \frac{s_{r2}}{\pi_r^2}\ \left(\lambda_0^2\mu_{-1}^2 - \mu_0^2\lambda_{-1}^2\right)\Big) \tag{10.50}$$

for each site r that corresponds to a primed vertex. We can evaluate this in a similar way to (10.42), to find, for example,

$$\begin{aligned}\langle e'_{-2}(\rho_1)e'_{-2}(\rho_2)C'(\rho_3)\rangle_{\text{tree}} &= \langle 0|e^{q_0}e'_{-2}(\rho_1)e'_{-2}(\rho_2)C'(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\ &= -\langle 12\rangle^4\,\frac{e'_{-2(1)}}{2p_1^0}\frac{e'_{-2(2)}}{2p_2^0}\frac{C'_{0(3)}}{2p_3^0}\frac{\partial^3}{(\partial P^0)^3}\delta^4(\Sigma\pi_r\bar{\pi}_r).\end{aligned} \tag{10.51}$$

So effectively, the contribution of a $V_{G'}(\rho_r)$ vertex operator to a tree amplitude can be found by replacing *each* unprimed wavefunction by a primed wavefunction times $-\frac{1}{2p_r^0}\frac{\partial}{\partial p_r^0}$.

Amplitudes involving V'_F vertices are more tedious to evaluate. As a guide for these methods, we can use the antiholomorphic amplitudes. For example, the $d = 0$ three-point coupling is

$$\begin{aligned}\langle A_1^{A_1}(\rho_1)A_1^{A_2}(\rho_2)\bar{C}'(\rho_3)\rangle_{\text{tree}} &= \int \langle 0|A_1^{A_1}(\rho_1)A_1^{A_2}(\rho_2)\bar{C}'(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_S d\gamma_M \\ = -\int \prod_{r=1}^{3} dk_r \frac{k_3^2}{k_1k_2}\ &\prod_{r=1}^{3} d\rho_r/d\gamma_S d\gamma_M\ \left(\frac{\delta^{A_1A_2}A_{(1)}A_{1(2)}\bar{C}'_{0(3)}}{(\rho_1-\rho_2)^2}\right) \\ \times \big(\bar{s}_3^a\ \langle 0|\prod_{c,r=1}^{2}\delta(\pi_r^c - k_r\lambda^c(\rho_r))\ &\delta(\pi_3^c - k_3\lambda^c(\rho_r))\ Y_a(\rho_3)|0\rangle \int\prod_a d\mu_0^a e^{i\sum_{r=1}^{3}k_r\bar{\pi}_{rb}\mu_0^b} \\ + is_3^{\dot{a}}\ \langle 0|e^{i\sum_{r=1}^{2}k_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}&e^{ik_3\bar{\pi}_{3\dot{b}}\mu^{\dot{b}}(\rho_3)}\,Y_{\dot{a}}(\rho_3)|0\rangle \\ \times\int\prod d\lambda_0^a\prod_{c,r=1}^{2}\delta(\pi_r^c - k_r\lambda^c(\rho_r))\ &\bar{s}_3^a\frac{\partial}{\partial\pi_3^a}\delta(\pi_3^c - k_3\lambda^c(\rho_3))\big) \\ = [12]^2\,\delta^{A_1A_2}A_{1(1)}A_{1(2)}\frac{\bar{C}'_{0(3)}}{2p_3^0}\frac{\partial}{\partial P^0}&\delta^4(\Sigma\pi_r\bar{\pi}_r),\end{aligned} \tag{10.52}$$

where we could evaluate $\langle 0| \prod_{c,r=1}^{2} \delta(\pi_r^c - k_r \lambda^c(\rho_r))\, \delta(\pi_3^c - k_3 \lambda^c(\rho_r))\; Y_a(\rho_3)|0\rangle$ by writing the delta functions $\prod_{c,r} \delta(\pi_r^c - k_r \lambda^c(\rho_r))$ as $\int \prod_{c,r} d\omega_{rc} e^{i \sum_{r=1}^{2} \omega_{rc}(k_r \lambda^c(\rho_r) - \pi_r^c)}$, using the commutator of $\lambda^a(\rho_r)$ with $Y_a(\rho_3)$, for $r = 1, 2$, and divide by the invariant measure (10.33). But we know the result, since it is the antiholomorphic form of (10.44), found by replacing $\pi_{ra}, \bar{\pi}_{r\dot{b}}$ with their conjugates $\bar{\pi}_{r\dot{a}}, \pi_{rb}$.

The non-vanishing three-point amplitudes involving the primed states of the dipoles from the sectors GGG', $GG'G'$, and $G'G'G'$, together with their helicity conjugates, are summarized in Tables 10.3 and 10.4.

$\langle A_{-1}^{A_1} A_{-1}^{A_2} C' \rangle = \langle 12 \rangle^2 \, \delta^{A_1 A_2} A_{-1(1)} A_{-1(2)} \frac{C'_{0(3)}}{2p_3^0} \frac{\partial}{\partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle A_1^{A_1} A_{-1}^{A_2} e'_{-2} \rangle = \frac{\langle 23 \rangle^4}{\langle 12 \rangle^2} \delta^{A_1 A_2} A_{1(1)} A_{-1(2)} \frac{e'_{-2(3)}}{2p_3^0} \frac{\partial}{\partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e_{-2} e_{-2} C' \rangle = -\langle 12 \rangle^4 e_{-2(1)} e_{-2(2)} \frac{C'_{0(3)}}{2p_3^0} \frac{\partial}{\partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e_{-2} e'_{-2} C \rangle = -\langle 12 \rangle^4 e_{-2(1)} \frac{e'_{-2(2)}}{2p_2^0} C_{0(3)} \frac{\partial}{\partial p_2^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e_{-2} e'_{-2} C' \rangle = -\langle 12 \rangle^4 e_{-2(1)} \frac{e'_{-2(2)}}{2p_2^0} \frac{C'_{0(3)}}{2p_3^0} \frac{\partial^2}{\partial p_2^0 \partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e'_{-2} e'_{-2} C \rangle = -\langle 12 \rangle^4 \frac{e'_{-2(1)}}{2p_1^0} \frac{e'_{-2(2)}}{2p_2^0} C_{0(3)} \frac{\partial^2}{\partial p_1^0 \partial p_2^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e'_{-2} e'_{-2} C' \rangle = -\langle 12 \rangle^4 \frac{e'_{-2(1)}}{2p_1^0} \frac{e'_{-2(2)}}{2p_2^0} \frac{C'_{0(3)}}{2p_3^0} \frac{\partial^3}{\partial p_1^0 \partial p_2^0 \partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$

Table 10.3: MHV Conformal supergravity couplings with primed states

$\langle A_1^{A_1} A_1^{A_2} \bar{C}' \rangle = [12]^2 \, \delta^{A_1 A_2} A_{1(1)} A_{1(2)} \frac{\bar{C}'_{0(3)}}{2p_3^0} \frac{\partial}{\partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle A_{-1}^{A_1} A_1^{A_2} e'_2 \rangle = \frac{[23]^4}{[12]^2} \delta^{A_1 A_2} A_{-1(1)} A_{1(2)} \frac{e'_{2(3)}}{2p_3^0} \frac{\partial}{\partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e_2 e_2 \bar{C}' \rangle = -[12]^4 e_{2(1)} e_{2(2)} \frac{\bar{C}'_{0(3)}}{2p_3^0} \frac{\partial}{\partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e_2 e'_2 \bar{C} \rangle = -[12]^4 e_{2(1)} \frac{e'_{2(2)}}{2p_2^0} \bar{C}_{0(3)} \frac{\partial}{\partial p_2^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e_2 e'_2 \bar{C}' \rangle = -[12]^4 e_{2(1)} \frac{e'_{2(2)}}{2p_2^0} \frac{\bar{C}'_{0(3)}}{2p_3^0} \frac{\partial^2}{\partial p_2^0 \partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e'_2 e'_2 \bar{C} \rangle = -[12]^4 \frac{e'_{2(1)}}{2p_1^0} \frac{e'_{2(2)}}{2p_2^0} \bar{C}_{0(3)} \frac{\partial^2}{\partial p_1^0 \partial p_2^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e'_2 e'_2 \bar{C}' \rangle = -[12]^4 \frac{e'_{2(1)}}{2p_1^0} \frac{e'_{2(2)}}{2p_2^0} \frac{\bar{C}'_{0(3)}}{2p_3^0} \frac{\partial^3}{\partial p_1^0 \partial p_2^0 \partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$

Table 10.4: $d = 0$ Conformal supergravity couplings with primed states

Chapter 11

Comparison with Known Conformal and Einstein Amplitudes

In this section, we extend our analysis of the three-point functions using canonical quantization, to N-point MHV tree amplitudes for unprimed vertex operators. The maximal helicity violating (MHV) amplitudes contain any two vertex operators of negative helicity, $e_{-2}, \bar{C}, A_{-1}$, and $N-2$ positive helicity vertex operators from the set e_2, C, A_1.

We will compute an amplitude for a specific choice of the two negative helicity states, and then discuss how this generalizes. We consider the $(d=1)$ amplitude for two negative helicity vertex operators, one of type G and one of type F, $\langle e_{-2}\bar{C}e_2 \ldots e_2 C \ldots C A_1 \ldots A_1\rangle_{\text{tree}}$. This has n positive helicity type F gravitons, m positive helicity type G scalars, and p positive helicity gluons . We denote the total number of vertices as $N = 2+n+m+p$. Inserting the expressions from Table C.1, we find

$$
\begin{aligned}
\int \langle 0|e^{q_0} \int dk_1 k_1 \lambda_{a_1}(\rho_1) \prod_{a=1}^{2} \delta(k_1\lambda^a(\rho_1) - {\pi_1}^a) e^{ik_1 \bar{\pi}_{1\dot{b}} \mu^{\dot{b}}(\rho_1)} \\
\times k_1^4 \psi^1(\rho_1)\psi^2(\rho_1)\psi^3(\rho_1)\psi^4(\rho_1)\partial\lambda^{a_1}(\rho_1) \\
\times i \int \frac{dk_2}{k_2^2} {\bar{\pi}_2}{}^{\dot{a}} \prod_{a=1}^{2} \delta(k_2\lambda^a(\rho_2) - {\pi_2}^a) e^{ik_2 \bar{\pi}_{2\dot{b}} \mu^{\dot{b}}(\rho_2)} k_2^4 \psi^1(\rho_2)\psi^2(\rho_2)\psi^3(\rho_2)\psi^4(\rho_2) Y_{\dot{a}}(\rho_2) \\
\times \prod_{j=3}^{n+2} i \int \frac{dk_j}{k_j^2} {\bar{\pi}_j}{}^{\dot{a}} \prod_{a=1}^{2} \delta(k_j\lambda^a(\rho_j) - {\pi_j}^a) e^{ik_j \bar{\pi}_{j\dot{b}} \mu^{\dot{b}}(\rho_j)} Y_{\dot{a}}(\rho_j) \\
\times \prod_{j=n+3}^{m+n+2} \int dk_j k_j \lambda_{a_j}(\rho_j) \prod_{a=1}^{2} \delta(k_j\lambda^a(\rho_j) - \pi_j^a) e^{ik_j \bar{\pi}_{j\dot{b}} \mu^{\dot{b}}(\rho_j)} \partial\lambda^{a_j}(\rho_j) \\
\times \prod_{j=m+n+3}^{N} \int \frac{dk_j}{k_j} \prod_{a=1}^{2} \delta(k_j\lambda^a(\rho_j) - \pi_j^a) e^{ik_j \bar{\pi}_{j\dot{b}} \mu^{\dot{b}}(\rho_j)} J^{A_j}(\rho_j)|0\rangle \prod_{r=1}^{N} d\rho_r / d\gamma_S d\gamma_M, \quad (11.1)
\end{aligned}
$$

where we have dropped the polarizations for convenience. It is useful to introduce the sets of indices: $\mathbf{n} = \{3, \ldots, n+2\}$, $\mathbf{m} = \{n+3, \ldots, m+n+2\}$, and $\mathbf{p} = \{m+n+3, \ldots, N\}$. To further emphasize the occurrence of gluon or graviton types, we define the larger sets $\mathbf{n}' = \{2, 3, \ldots, n+2\}$ and $\mathbf{m}' = \{1, n+3, \ldots, m+n+2\}$. From the following formula presented below, we can see these sets will be useful when considering amplitudes having a more complicated ordering of vertex operators. We rewrite (11.1) as

$$(i)^{n+1}\int\prod_{r=1}^{N}dk_r d\rho_r/d\gamma_S d\gamma_M \prod_{a,r}\delta(k_r\lambda^a(\rho_r)-\pi_r{}^a)\prod_{j\in\mathbf{n}'}\left(\frac{1}{k_j}\right)^2\prod_{j\in\mathbf{m}'}(k_j)\prod_{j\in\mathbf{p}}\left(\frac{1}{k_j}\right)$$
$$\times(k_1k_2)^4\langle 0|e^{q_0}\psi^1(\rho_1)\psi^1(\rho_2)\psi^2(\rho_1)\psi^2(\rho_2)\psi^3(\rho_1)\psi^3(\rho_2)\psi^4(\rho_1)\psi^4(\rho_2)|0\rangle$$
$$\times\langle 0|e^{q_0}\prod_{j\in\mathbf{m}'}\lambda_a(\rho_j)\partial\lambda^a(\rho_j)|0\rangle\ \langle 0|\prod_{j\in\mathbf{p}}J^{A_j}(\rho_j)|0\rangle$$
$$\times\langle 0|e^{q_0}e^{ik_1\bar{\pi}_{1\dot{b}}\mu^{\dot{b}}(\rho_1)}e^{ik_2\bar{\pi}_{2\dot{b}}\mu^{\dot{b}}(\rho_2)}\bar{\pi}_2{}^{\dot{a}}Y_{\dot{a}}(\rho_2)\prod_{j\in\mathbf{n}}\left(e^{ik_j\bar{\pi}_{j\dot{b}}\mu^{\dot{b}}(\rho_j)}\bar{\pi}_j^{\dot{a}}Y_{\dot{a}}(\rho_j)\right)$$
$$\times\prod_{j\in\mathbf{m}}\left(e^{ik_j\bar{\pi}_{j\dot{b}}\mu^{\dot{b}}(\rho_j)}\right)\prod_{j\in\mathbf{p}}\left(e^{ik_j\bar{\pi}_{j\dot{b}}\mu^{\dot{b}}(\rho_j)}\right)|0\rangle. \tag{11.2}$$

Many simplifications happen at this stage. With

$$\langle 0|e^{q_0}\psi^1(\rho_1)\psi^1(\rho_2)|0\rangle=(\rho_1-\rho_2)\langle 0|e^{q_0}\psi^1_{-1}\psi^1_0|0\rangle=(\rho_1-\rho_2), \tag{11.3}$$

four factors of $\rho_1-\rho_2$ come from the second line. Evaluating the λ term, we find

$$\langle 0|e^{q_0}\prod_{j\in\mathbf{m}'}\lambda_{a_j}(\rho_j)\partial\lambda^{a_j}(\rho_j)|0\rangle=\int\prod_a d^2\lambda^a(\det\lambda)^{m+1}, \tag{11.4}$$

where $\det\lambda=\lambda^1_0\lambda^2_{-1}-\lambda^2_0\lambda^1_{-1}$, as in Chapter 10. We use a current algebra contribution [59]

$$\langle 0|\prod_{j\in\mathbf{p}}J^{A_j}(\rho_j)|0\rangle=f^{A_{m+n+3}\cdots A_N}\prod_{j\in\mathbf{p}}\frac{1}{\rho_j-\rho_{j+1}}, \tag{11.5}$$

with $\rho_{N+1}\equiv\rho_{m+n+3}$. In what follows, we denote $f^{A_{m+n+3}\ldots A_N}=f^{A\ldots A}$. This is merely simplification of notation, as the group indices add no new information not contained in the denominator. We note, for computing MHV amplitudes containing negative helicity gluons, that the form (11.5) remains the same with the set $\mathbf{p}$ replaced by the total set of gluons $\mathbf{p}'$.

The last expectation value in (11.2) is equal to

$$(-i)^{n+1}\,(\det\lambda)^2\delta^4\left(\Sigma\pi_r\bar{\pi}_r\right)\prod_{x\in\mathbf{n}'}\sum_{y=1,y\neq x}^{N}k_y\frac{[xy]}{\rho_y-\rho_x}, \tag{11.6}$$

where now $\delta^4\left(\Sigma\pi_r\bar{\pi}_r\right)\equiv\prod_{\dot{a},b}\delta\left(\Sigma_{r=1}^N\pi_r^b\bar{\pi}_{r\dot{a}}\right)$. We integrate the k_r's using $\delta(k_r-\pi_r^1/\lambda^1(\rho_r))$, and evaluate the amplitude (11.2) to obtain

$$\begin{aligned}&\delta^4(\Sigma\pi_r\bar{\pi}_r)\int\prod_{r=1}^N d\rho_r\prod_a d^2\lambda^a/d\gamma_S\gamma_M\prod_{r=1}^N\delta({\pi_r}^2-\frac{\lambda^2(\rho_r)}{\lambda^1(\rho_r)}{\pi_r}^1)\prod_{r=1}^N\frac{1}{\lambda^1(\rho_r)}\\&\times\left(\frac{{\pi_1}^1}{\lambda^1(\rho_1)}\frac{{\pi_2}^1}{\lambda^1(\rho_2)}\right)^4(\rho_1-\rho_2)^4\prod_{j\in\mathbf{n}'}\left(\frac{\lambda^1(\rho_j)}{{\pi^1}_j}\right)^2\prod_{j\in\mathbf{m}'}\left(\frac{{\pi_j}^1}{\lambda^1(\rho_j)}\right)\prod_{j\in\mathbf{p}}\left(\frac{\lambda^1(\rho_j)}{{\pi_j}^1}\right)\\&\times f^{A\cdots A}\prod_{j\in\mathbf{p}}\left(\frac{1}{\rho_j-\rho_{j+1}}\right)(\det\lambda)^{m+3}\prod_{x\in\mathbf{n}'}\sum_{y=1,y\neq x}^N\frac{{\pi_y}^1}{\lambda^1(\rho_y)}\frac{[xy]}{\rho_y-\rho_x}\end{aligned} \tag{11.7}$$

We define $\zeta_r=\frac{\lambda^2(\rho_r)}{\lambda^1(\rho_r)}$ and change variables from ρ_r to ζ_r. The identification

$$\sum_{y=1,y\neq x}^N\frac{{\pi_y}^1}{\lambda^1(\rho_y)}\frac{[xy]}{\rho_y-\rho_x}=\frac{\det\lambda}{(\lambda^1(\rho_x))^3}\sum_{y=1,y\neq x}^N\frac{{\pi_y}^1[xy]}{\zeta_y-\zeta_x} \tag{11.8}$$

follows from

$$\sum_{y\neq x}\frac{\pi_y^1[xy]}{\lambda^1(\rho_y)(\rho_y-\rho_x)}=\frac{1}{\lambda^1(\rho_x)}\sum_{y\neq x}\frac{\pi_y^1[xy]}{(\rho_y-\rho_x)}, \tag{11.9}$$

using $\sum_y\frac{\pi_y^1\bar{\pi}_{y\dot{b}}}{\lambda^1(\rho_y)}=0$, which is provided by the factor $\delta^4(\Sigma\pi_r\bar{\pi}_r)$ in (11.7), in view of the equality (10.40). To implement the change of variables, we have $\zeta_r-\zeta_j=\frac{(\rho_r-\rho_j)\det\lambda}{\lambda^1(\rho_r)\lambda^1(\rho_j)}$,

$d\zeta = \frac{\det\lambda}{\lambda^1(\rho)^2} d\rho$, so (11.7) is

$$\delta^4(\Sigma\pi_r\bar{\pi}_r) \int \prod_{r=1}^{N} d\zeta_r \prod_a d^2\lambda^a/d\gamma_S d\gamma_M \,(\det\lambda)^{-2} \prod_{r=1}^{N} \delta({\pi_r}^2 - \zeta_r\pi_r^1)\left(\pi_1{}^1\pi_2{}^1(\zeta_1-\zeta_2)\right)^4$$
$$\times \prod_{j\in\mathbf{n}'}\left(\frac{1}{\pi_j{}^1}\right)^2 \prod_{j\in\mathbf{m}'}(\pi_j{}^1)\prod_{j\in\mathbf{p}}\left(\frac{1}{\pi_j{}^1}\right) f^{A\cdots A}\prod_{j\in\mathbf{p}}\left(\frac{1}{\zeta_j-\zeta_{j+1}}\right)\prod_{x\in\mathbf{n}'}\sum_{y=1,y\neq x}^{N}\pi_y{}^1\frac{[xy]}{\zeta_y-\zeta_x} \tag{11.10}$$

We identify $d\gamma_S d\gamma_M = d^2\lambda^a(\det\lambda)^{-2}$ and do the ζ_r integrations. Since

$$\prod_{x\in\mathbf{n}'}\left(\frac{1}{\pi_x{}^1}\right)^3\sum_{y=1,y\neq x}^{N}\pi_y{}^1\frac{[xy]}{\zeta_y-\zeta_x} = \prod_{x\in\mathbf{n}'}\sum_{y=1,y\neq x}^{N}\frac{(\pi_y^1)^2}{(\pi_x^1)^2}\frac{[xy]}{\langle xy\rangle}, \tag{11.11}$$

(11.11) can be reexpressed as [35]

$$\prod_{x\in\mathbf{n}'}\sum_{y=1,y\neq x}^{N}\frac{\langle y\xi\rangle^2}{\langle x\xi\rangle^2}\frac{[xy]}{\langle xy\rangle}, \tag{11.12}$$

which is independent of π_ξ,$\bar{\pi}_\xi$, to obtain the result

$$\langle e_{-2}\bar{C}e_2\cdots e_2C\cdots CA_1\ldots A_1\rangle$$
$$= \delta^4(\Sigma\pi_r\bar{\pi}_r)\langle 12\rangle^4 f^{A\cdots A}\prod_{j\in\mathbf{p}}\frac{1}{\langle j,j+1\rangle}\prod_{i\in\mathbf{n}'}\sum_{j=1,j\neq i}^{N}\frac{\langle j\xi\rangle^2}{\langle i\xi\rangle^2}\frac{[ij]}{\langle ij\rangle}. \tag{11.13}$$

The amplitude is independent of the order of the positive helicity states, as this corresponds merely to changing the position of the $Y_{\dot{a}}$ fields, and does not affect (11.6). To generalize our expression for any two negative helicity states, it is useful to identify pieces common to all amplitudes: the two negative helicity states in any position ρ_r, ρ_s will give the factor of $\langle rs\rangle^4$, all gluon vertices contribute to $f^{A\cdots A}\prod_{j\in\mathbf{p}'}\frac{1}{\langle j,j+1\rangle}$, defined in (11.5), and the type F vertex operators contribute to the product of sums. The type G vertex operators provide factors of $\det\lambda$, and leave no further mark on the amplitude.

Our answer (11.13) thus becomes the Berkovits Witten formula [35], which they found from a path integral formulation, and where we have absorbed a factor $(-i)^F$ in the definition of the vertex operators V_F.

To better visualize conformal gravity amplitudes better, we use (11.13) to study the conformal four-graviton tree amplitude

$$
\begin{aligned}
&\langle e_{-2}(\rho_1)e_{-2}(\rho_2)e_2(\rho_3)e_2(\rho_4)\rangle_{CG} = \langle 12\rangle^4 \prod_{j=3,4}\sum_{k\neq j}\frac{[jk]\langle k\xi\rangle^2}{\langle jk\rangle\langle j\xi\rangle^2} \\
&= -\frac{\langle 12\rangle^4[32]\langle 21\rangle\left(\langle 43\rangle\langle 21\rangle-\langle 23\rangle\langle 41\rangle\right)[42]\langle 21\rangle\left(\langle 34\rangle\langle 21\rangle-\langle 24\rangle\langle 31\rangle\right)}{\langle 31\rangle^2\langle 41\rangle^2\langle 34\rangle^2\langle 23\rangle\langle 42\rangle} \quad (\text{choose } \xi = 1) \\
&= \frac{\langle 12\rangle^4[34]^4}{(s_{12})^2} \qquad \text{using the identity } \langle 43\rangle\langle 21\rangle-\langle 23\rangle\langle 41\rangle = \langle 13\rangle\langle 24\rangle \\
&= \frac{s_{23}s_{24}}{s_{12}}\,\langle e_{-2}(1)e_{-2}(2)e_2(3)e_2(4)\rangle_{Einstein}, \qquad (11.14)
\end{aligned}
$$

which has fewer poles than Einstein gravity, since the Berends-Giele-Kuijf expression [45] for Einstein gravity tree amplitudes as a product of Yang-Mills trees is

$$
\begin{aligned}
\langle e_{-2}(1)e_{-2}(2)e_2(3)e_2(4)\rangle_{Einstein} &= s_{12}\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,\frac{\langle 12\rangle^3}{\langle 24\rangle\langle 43\rangle\langle 31\rangle} \\
&= \frac{1}{s_{12}s_{23}s_{24}}\langle 12\rangle^4[34]^4. \qquad (11.15)
\end{aligned}
$$

We can reproduce the conformal gravity four-point function (11.14) from tree level exchange of the scalar field C

$$\langle e_{-2}(\rho_1)e_{-2}(\rho_2)e_2(\rho_3)e_2(\rho_4)\rangle_{CG} = \langle 12\rangle^4\frac{1}{(s_{12})^2}[34]^4 \qquad (11.16)$$

corresponding to the product of the three-point trees $\langle e_{-2}(\rho_1)e_{-2}(\rho_2)C(\rho)\rangle = \langle 12\rangle^4$ and

$\langle \bar{C}(\rho)e_2(\rho_3)e_2(\rho_4)\rangle = [34]^4$, times the conformal propagator $\frac{1}{(p^2)^2}$, where $p^2 = s_{12}$.

Chapter 12

N-Point Tree Amplitudes for Primed and Unprimed Vertices

Finally we turn to the N-point MHV scattering amplitudes containing both primed and unprimed vertices. To begin, consider two negative helicity gluons, A_{-1} and n G' scalars, C'_0. The total number of vertices is $N = 2 + n$. The set of primed vertices is $\mathbf{n} = \{3, \ldots, N\}$. To compute $\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C'_0(\rho_3)\ldots C'_0(\rho_N)\rangle_{\text{tree}}$, use the vertex operators from Table C.1:

$$
\begin{aligned}
&\int \langle 0| e^{q_0} \int \frac{dk_1}{k_1} \prod_a \delta\left(k_1\lambda^a(\rho_1) - {\pi_1}^a\right) e^{ik_1\bar{\pi}_{1\dot{b}}\mu^{\dot{b}}(\rho_1)} k_1^4 \psi^1(\rho_1)\psi^2(\rho_1)\psi^3(\rho_1)\psi^4(\rho_1) A_{-1(1)} J^{A_1}(\rho_1) \\
&\quad\times \int \frac{dk_2}{k_2} \prod_a \delta\left(k_2\lambda^a(\rho_2) - {\pi_2}^a\right) e^{ik_2\bar{\pi}_{2\dot{b}}\mu^{\dot{b}}(\rho_2)} k_2^4 \psi^1(\rho_2)\psi^2(\rho_2)\psi^3(\rho_2)\psi^4(\rho_2) A_{-1(2)} J^{A_2}(\rho_2) \\
&\quad\times \prod_{j\in \boldsymbol{n}} \Bigg(i \int dk_j k_j \prod_a \delta\left(k_j\lambda^a(\rho_j) - {\pi^a}_j\right) e^{ik_j\bar{\pi}_{j\dot{b}}\mu^{\dot{b}}(\rho_j)} \\
&\qquad\quad \times \left[s_{j\dot{a}}\partial\mu^{\dot{a}}(\rho_j) - \bar{s}_{ja}s_{j\dot{a}}\mu^{\dot{a}}(\rho_j)\partial\lambda^a(\rho_j)\right] C'_{0(j)} \Bigg) |0\rangle \prod_{r=1}^{N} d\rho_r/d\gamma_S d\gamma_M,
\end{aligned}
\tag{12.1}
$$

which yields

$$i^n \int \prod_{r=1}^{N} dk_r d\rho_r k_r \prod_{a,r} \delta\left(k_r \lambda^a(\rho_r) - \pi_r{}^a\right) (\rho_1 - \rho_2)^4 (k_1 k_2)^4 \prod_a d^2\lambda^a / d\gamma_S d\gamma_M$$
$$\times \frac{-\delta^{A_1 A_2}}{(\rho_1 - \rho_2)^2} \left(\frac{1}{k_1 k_2}\right)^2 A_{-1(1)} A_{-1(2)} C'_{0(3)} \cdots C'_{0(N)}$$
$$\times \langle 0| e^{q_0} \prod_{j \in \boldsymbol{n}} \left(\frac{s_{j\dot{a}}}{k_j} \partial \mu^{\dot{a}}(\rho_j) - \bar{s}_{ja} s_{j\dot{a}} \mu^{\dot{a}}(\rho_j) \partial \lambda^a(\rho_j) \right) e^{i \sum_r k_r \bar{\pi}_{r\dot{b}} \mu^{\dot{b}}(\rho_j)} |0\rangle. \qquad (12.2)$$

We evalute the expectation value in (12.2) as a sequence of derivatives, as in (10.51), and find

$$\langle A_{-1}^{A_1}(\rho_1)_{-1}^{A_2}(\rho_2) C'_0(\rho_3) \ldots C'_0(\rho_N) \rangle_{\text{tree}}$$
$$= -\langle 12 \rangle^2 \delta^{A_1 A_2} A_{-1(1)} A_{-1(2)} C'_{0(3)} \ldots C'_{0(N)} \prod_{j \in \boldsymbol{n}} \left[-\frac{1}{2p_j^0} \frac{\partial}{\partial p_j^0} \right] \delta^4(\Sigma \pi_r \bar{\pi}_r). \quad (12.3)$$

Clearly this same form holds for

$$\langle e'_{-2}(\rho_1) e'_{-2}(\rho_2) C'_0(\rho_3) \ldots C'_0(\rho_N) \rangle_{\text{tree}}$$
$$= \langle 12 \rangle^4 e'_{-2(1)} e'_{-2(2)} C'_{0(3)} \ldots C'_{0(N)} \prod_{j \in N} \left[-\frac{1}{2p_j^0} \frac{\partial}{\partial p_j^0} \right] \delta^4(\Sigma \pi_r \bar{\pi}_r), \qquad (12.4)$$

and for any combination of these type G primed and unprimed states, with the product then taken over the primed sites.

N-point functions with type $F$and F' vertices are more varied to track. For the four-point function, we can use the anti-holmorphic form of $\langle e'_{-2}(\rho_1) e'_{-2}(\rho_2) e_2(\rho_3) e_2(\rho_4) \rangle_{\text{tree}}$ to evaluate the $F'F'GG$ amplitude $\langle e'_2(\rho_1) e'_2(\rho_2) e_{-2}(\rho_3) e_{-2}(\rho_4) \rangle_{\text{tree}}$.

Chapter 13

Conclusion

We identify the Yangian structure for the SU$(2|3)$ spin chain Hamiltonian, which is the one-loop dilatation operator for a subset of states in the marginally deformed Yang-Mills theory. Like the parent case of PSU$(2,2|4)$, we check that the Yangian commutation relations hold to one-loop. With the twisted R-matrix of the Yang-Baxter equation, supplied by Beisert and Roiban, we compute the twisted coproducts using the Reshetikhin formalism. This twisted coproduct leaves a residual unbroken symmetry.

We find that useful identities derived in the undeformed theory have a twisted analog. We explicitly calculate the twisted quadratic Casimir acting on two-particle states, and show it is equivalent to the deformed Hamiltonian of the theory.

We go on to show that the residual symmetries found above correspond to the standard, undeformed coproduct. The broken symmetries have phases associated with their coproducts, so they are twisted. In general, to find a given residual symmetry, one could start by assuming untwisted coproducts for the desired generators.

For chains of length larger than two, the twisted coproducts give a formalism that maintains 'edge' effects and therefore can be used to check the Yangian symmetry extrapolated to one-loop. Although higher loops in the SU$(2|3)$ sector have dynamical lengths, our argument suggests the Yangian structure will survive to all orders, and therefore we expect to find the SU(2|3) Yangian symmetry with twisted coproducts in the world-sheet of the dual string theory.

Although the entire SU(2|3) Yangian algebra is still present in the deformed theories, and is responsible for its integrability, the symmetries of the deformed field theories are residual groups which result from the twisted coproducts. These subgroups $SU(2)\times U(1)^3$ in Case 1, and $SU(2|1)\times U(1)^2$ in Case 2 correspond to the unbroken subgroups of $SU(2,2|1)\times U(1)^2$ in the $\mathcal{N}=1$ superconformal deformed gauge field theory, that survive in its SU(2|3) sector, which we consider in this paper.

The twisted coproduct has provided a mechanism for maintaining integrability in a theory while breaking some of its initial symmetry. This procedure might be useful in formulating integrable versions of even smaller symmetries, possibly $\mathcal{N}=0$ Yang-Mills.

In the second part of this dissertation we consider the open twistor string. This is a description of $\mathcal{N}=4$ Yang-Mills field theory coupled to $\mathcal{N}=4$ conformal supergravity. We derive a canonical quantization representation for all conformal supergravity vertices outlined in [35]. We use a subset of these vertex operators (listed in Appendix C) to calculate non-trivial n-point tree amplitudes. These include the vertices for the 'dipole' graviton states. Little work has been done with these states as they are not eigenstates of the translation generator.

We compute three-point tree amplitudes. We find the form of the amplitudes that include the dipole states to follow the same general form as the unprimed, non-dipole states, however they can include a derivative on the momentum conserving delta function. We then compute general N-point tree level MHV amplitudes involving non-dipole states, using our canonical methods. Finally, we compute N-point MHV tree amplitudes involving both types of graviton states. These tree amplitudes are hard to access in field theory calculations of conformal supergravity, but are fairly straightforward in the string theory formulation.

Appendix A

Single Index Basis for Yangian Generators

It is sometimes convenient to chose a single index basis for the twenty-four generators of SU(2|3). The single index basis for the twelve even generators take the representation:

SU(3)	**SU(2)**	**U(1)**
$J^1 = R^1{}_2 + R^2{}_1$	$J^9 = L^1{}_2 + L^2{}_1$	$J^{12} = D$
$J^2 = i(R^1{}_2 - R^2{}_1)$	$J^{10} = i(L^1{}_2 - L^2{}_1)$	
$J^3 = R^1{}_1 - R^2{}_2$	$J^{11} = L^1{}_1 - L^2{}_2$	
$J^4 = R^1{}_3 + R^3{}_1$		
$J^5 = i(R^1{}_3 - R^3{}_1)$		
$J^6 = R^2{}_3 + R^3{}_2$		
$J^7 = i(R^2{}_3 - R^3{}_2)$		
$J^8 = R^1{}_1 + R^2{}_2 - 2R^3{}_3$		

Table A.1: Even symmetry generators of SU(2|3).

The twelve odd generators have the representation:

$J^{13} = S^1{}_1 + Q^1{}_1$	$J^{19} = S^2{}_1 + Q^1{}_2$
$J^{14} = i(S^1{}_1 - Q^1{}_1)$	$J^{20} = i(S^2{}_1 + Q^1{}_2)$
$J^{15} = S^1{}_2 + Q^2{}_1$	$J^{21} = S^2{}_2 + Q^2{}_2$
$J^{16} = i(S^1{}_2 - Q^2{}_1)$	$J^{22} = i(S^2{}_2 + Q^2{}_2)$
$J^{17} = S^1{}_3 + Q^3{}_1$	$J^{23} = S^2{}_3 + Q^3{}_2$
$J^{18} = i(S^1{}_3 - Q^3{}_1)$	$J^{24} = i(S^2{}_3 + Q^3{}_2)$

Table A.2: Odd symmetry generators of SU(2|3)

The metric is symmetric (and diagonal) in the even regions and antisymmetric in the odd regions.

$$g_{(1)(1)} = g_{(2)(2)} = g_{(3)(3)} = g_{(4)(4)} = g_{(5)(5)} = g_{(6)(6)} = g_{(7)(7)} = g_{(8)(8)} = -\tfrac{1}{4},$$

$$g_{(9)(9)} = g_{(10)(10)} = g_{(11)(11)} = \tfrac{1}{4},$$

$$-g_{(13)(14)} = g_{(14)(13)} = -g_{(15)(16)} = g_{(16)(15)} = \cdots = g_{(22)(21)} = -g_{(23)(24)} = g_{(24)(23)} = \tfrac{1}{4i}. \tag{A.1}$$

The SU(2|3) algebra obeys the commutation $[J^A, J^B] = f^{ABC} g_{CD} J^D$. The structure constants are totally antisymmetric with an additional minus sign under the interchange of two odd indices.

$$
\begin{aligned}
& f^{(1)(2)(3)} = -f^{(9)(10)(11)} = -8i \\
& f^{(1)(4)(7)} = f^{(1)(6)(5)} = f^{(2)(4)(7)} = f^{(2)(5)(7)} = f^{(3)(4)(5)} = f^{(3)(7)(6)} = -4i \\
& f^{(4)(5)(8)} = f^{(6)(7)(8)} = -12i \\
& f^{(1)(13)(15)} = f^{(1)(14)(16)} = f^{(1)(19)(21)} = f^{(1)(20)(22)} = -4 \\
& f^{(2)(13)(16)} = -f^{(2)(14)(15)} = f^{(2)(19)(22)} = -f^{(2)(20)(21)} = -4 \\
& f^{(3)(13)(13)} = f^{(3)(14)(14)} = -f^{(3)(15)(15)} = -f^{(3)(16)(16)} = -4 \\
& f^{(3)(19)(19)} = f^{(3)(20)(20)} = -f^{(3)(21)(21)} = -f^{(3)(22)(22)} = -4 \\
& f^{(4)(13)(17)} = f^{(4)(14)(18)} = f^{(4)(19)(23)} = f^{(4)(20)(24)} = -4 \\
& f^{(5)(13)(18)} = f^{(5)(14)(17)} = f^{(5)(19)(24)} = f^{(5)(20)(23)} = -4 \\
& f^{(6)(15)(17)} = f^{(6)(16)(18)} = f^{(6)(21)(23)} = f^{(6)(22)(24)} = -4 \\
& f^{(7)(15)(18)} = -f^{(7)(16)(17)} = f^{(7)(21)(24)} = -f^{(7)(22)(23)} = -4 \\
& f^{(8)(13)(13)} = f^{(8)(14)(14)} = f^{(8)(15)(15)} = f^{(8)(16)(16)} = -4 \\
& f^{(8)(19)(19)} = f^{(8)(20)(20)} = f^{(8)(21)(21)} = f^{(8)(22)(22)} = -4 \\
& f^{(8)(17)(17)} = f^{(8)(18)(18)} = f^{(8)(23)(23)} = f^{(8)(24)(24)} = 8 \\
& f^{(9)(13)(19)} = f^{(9)(14)(20)} = f^{(9)(15)(21)} = f^{(9)(16)(22)} = f^{(9)(17)(23)} = f^{(9)(18)(24)} = 4 \\
& f^{(10)(13)(20)} = -f^{(10)(14)(19)} = f^{(10)(15)(22)} = -f^{(10)(16)(21)} = f^{(10)(17)(24)} = -f^{(10)(18)(23)} = -4 \\
& f^{(11)(13)(13)} = f^{(11)(14)(14)} = f^{(11)(15)(15)} = f^{(11)(16)(16)} = f^{(11)(17)(17)} = f^{(11)(18)(18)} = 4 \\
& f^{(11)(19)(19)} = f^{(11)(20)(20)} = f^{(11)(21)(21)} = f^{(11)(22)(22)} = f^{(11)(23)(23)} = f^{(11)(24)(24)} = -4 \\
& f^{(12)(13)(13)} = f^{(12)(14)(14)} = f^{(12)(15)(15)} = f^{(12)(16)(16)} = f^{(12)(17)(17)} = f^{(12)(18)(18)} = 2 \\
& f^{(12)(19)(19)} = f^{(12)(20)(20)} = f^{(12)(21)(21)} = f^{(12)(22)(22)} = f^{(12)(23)(23)} = f^{(12)(24)(24)} = 2.
\end{aligned}
\tag{A.2}
$$

Appendix B

Double Index Basis for Yangian Generators

The generators can be written in a double index notation [55, 32]. For a general superalgebra, we can define matrices $(E_{AB})_{ij} = \delta_{Ai}\delta_{Bj}$ which satisfy $[E_{AB}, E_{CD}] = \delta_{CB}E_{AD} - \delta_{AD}E_{CB}$. For SU(2|3) ,

$$\begin{aligned}
&[R^a{}_b, R^c{}_d] = \delta^a_d R^c{}_b - \delta^c_b R^a{}_d, \quad [R^a{}_b, Q^c{}_\gamma] = -\delta^c_b Q^a{}_\gamma + \frac{1}{3}\delta^a_b Q^c{}_\gamma, \\
&[R^a{}_b, S^\gamma{}_c] = \delta^a_c S^\gamma{}_b - \frac{1}{3}\delta^a_b S^\gamma{}_c, \quad [L^\alpha{}_\beta, L^\gamma{}_\delta] = \delta^\alpha_\delta L^\gamma{}_\beta - \delta^\gamma{}_\beta L^\alpha{}_\delta, \\
&[L^\alpha{}_\beta, Q^c{}_\gamma] = \delta^\alpha_\gamma Q^c_\beta - \frac{1}{2}\delta^\alpha_\beta Q^c{}_\gamma, \quad [L^\alpha{}_\beta, S^\gamma{}_c] = -\delta^\gamma_\beta S^\alpha_c + \frac{1}{2}\delta^\alpha_\beta S^\gamma_c, \\
&\quad [D, Q^a{}_\alpha] = +\frac{1}{2}Q^a{}_\alpha, \quad [D, S^\alpha{}_a] = -\frac{1}{2}S^\alpha_a, \\
&\quad \left\{Q^a{}_\alpha, S^\beta{}_b\right\} = \delta^\beta_\alpha R^a{}_b + \delta^a_b L^\beta{}_b + \frac{1}{3}\delta^a_b \delta^\beta_\alpha D. \qquad \text{(B.1)}
\end{aligned}$$

We transform between our SU(2|3) generators and the general matrices defined above by

$$\begin{aligned}
&R^a{}_b = E_{ba} - \frac{1}{3}\delta^a_b E_{cc}, \quad L^\alpha{}_\beta = E_{\beta\alpha} - \frac{1}{2}\delta^\alpha_\beta E_{\gamma\gamma}, \\
&S^\gamma{}_c = E_{c\gamma}, \quad Q^c{}_\gamma = E_{\gamma c}, \quad D = E_{cc} + \frac{3}{2}E_{\gamma\gamma}, \qquad \text{(B.2)}
\end{aligned}$$

where repeated indices are summed over.

This is an equivalent representation to the single index basis. For example, we give the defining relations [55]

$$\begin{aligned}
\left[J^A{}_B, J^C{}_D\right] &= \delta^C_B J^A{}_D - \delta^A_D J^C{}_B, \\
\left[J^A{}_B, \hat{J}^C{}_D\right] &= \delta^C_B \hat{J}^A{}_D - \delta^A_D \hat{J}^C{}_B, \\
\left[J^A{}_B, \left[\hat{J}^C{}_D, \hat{J}^E{}_F\right]\right] - \left[\hat{J}^A{}_B, \left[J^C{}_D, \hat{J}^E{}_F\right]\right] &= \tfrac{h^2}{4} \textstyle\sum_{P,Q} \big(\left[J^A{}_B, \left[J^C{}_P J^P{}_D, J^E{}_Q J^Q{}_F\right]\right] \\
&\quad - \left[J^A{}_P J^P{}_B, \left[J^C{}_D, J^E{}_Q J^Q{}_F\right]\right] \big) .
\end{aligned} \tag{B.3}$$

Deforming coproducts of the infinite Yangian algebra, a combination of the above representation is used:

$$\begin{aligned}
&J^A{}_B = t^{(0)}_{AB}, \\
&\hat{J}^A{}_B = t^{(1)}_{AB} - \frac{h}{2} \sum_D t^{(0)}_{AD} t^{(0)}_{DB}.
\end{aligned} \tag{B.4}$$

The coproducts (7.6) use the above $t^{(0)}_{AB}$'s for the deformation [32].

Appendix C

A Subset of Vertex Operators for Calculating Tree Amplitudes

We will focus on amplitudes involving a subset of the vertex operators. We define $\psi^a(\rho) \equiv \psi^a$.

V_F	$e_2(\rho) = i\int dk\, k^{-2}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}\,\bar{\pi}^{\dot{a}}Y_{\dot{a}}(\rho)\,e_2$
	$\bar{C}(\rho) = i\int dk\, k^{-2}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}\,k^4\psi^1\psi^2\psi^3\psi^4\,\bar{\pi}^{\dot{a}}Y_{\dot{a}}(\rho)\,\bar{C}_0$
V_G	$C(\rho) = \int dk\, k\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}\,\lambda_a(\rho)\partial\lambda^a(\rho)\,C_0$
	$e_{-2}(\rho) = \int dk\, k\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}\,k^4\psi^1\psi^2\psi^3\psi^4\,\lambda_a(\rho)\partial\lambda^a(\rho)\,e_{-2}$
$V_{F'}$	$e_2'(\rho) = \int dk\, k^{-2}\Big[\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)\bar{s}^aY_a(\rho) + i\bar{s}^b\big(\frac{\partial}{\partial\pi^b}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)\big)s^{\dot{a}}Y_{\dot{a}}(\rho)\Big]\,e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}e_2'$
	$\bar{C}'(\rho) = \int dk\, k^{-2}\Big[\bar{s}^aY_a(\rho)\,\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a) + i\bar{s}^b\big(\frac{\partial}{\partial\pi^b}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)\big)s^{\dot{a}}Y_{\dot{a}}(\rho)\Big]\,e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}k^4\psi^1\psi^2\psi^3\psi^4\,\bar{C}_0'$
$V_{G'}$	$C'(\rho) = i\int dk\, k\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a) \times\big[k^{-1}s_{\dot{a}}\partial\mu^{\dot{a}}(\rho) - s_{\dot{a}}\mu^{\dot{a}}(\rho)\bar{s}_a\partial\lambda^a(\rho)\big]\,e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}C_0'$
	$e_{-2}'(\rho) = i\int dk\, k\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a) \times\big[k^{-1}s_{\dot{a}}\partial\mu^{\dot{a}}(\rho) - s_{\dot{a}}\mu^{\dot{a}}(\rho)\bar{s}_a\partial\lambda^a(\rho)\big]\,e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}\,k^4\psi^1\psi^2\psi^3\psi^4\,e_{-2}'$
V_Φ	$A_1^A(\rho) = \int dk\, k^{-1}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}\,A_1\;J^A(\rho)$
	$A_{-1}^A(\rho) = \int dk\, k^{-1}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}k^4\psi^1\psi^2\psi^3\psi^4\,A_{-1}\,J^A(\rho)$

Table C.1: A subset of the vertex operators: conformal gravitons, scalars and gluons

Bibliography

[1] J. Ihry *Yangians in Deformed Super Yang-Mills Theories, JHEP* **0804**, 051 (2008) [arXiv:0802.3644]

[2] L. Dolan and J. N. Ihry, *Conformal Supergravity Tree Amplitudes from Open Twistor String Theory*, [arXiv:0811.1341].

[3] R. G. Leigh and M. J. Strassler, *Exactly Marginal Operators and Duality in Four-Dimensional N=1 Supersymmetric Gauge Theory, Nucl. Phys.* **B447** (1995) 95 [hep-th/9503121]

[4] O. Lunin and J. Maldacena, *Deforming Field Theories with U(1) x U(1) Global Symmetry and their Gravity Duals, JHEP* **05** (2005) 033 [hep-th/0502086]

[5] S. Frolov, *Lax Pair for Strings in Lunin-Maldacena Background, JHEP* **05** (2005) 069 [hep-th/0503201]

[6] R. Roiban, *On Spin Chains and Field Theories, JHEP* **09** (2004) 023 [hep-th/0312218]

[7] S. A. Frolov, R. Roiban and A. A. Tseytlin, *Gauge-String Duality for (non)Supersymmetric Deformations of $N = 4$ Super Yang-Mills Theory, Nucl. Phys.* **B731** (2005) 1 [hep-th/0507021]

[8] D. Berenstein and S. A. Cherkis *Deformations of $N = 4$ SYM and Integrable Spin Chain Models, Nucl. Phys.* **B702** (2004) 49 [hep-th/0405215]

[9] N. Beisert and R. Roiban, *Beauty and the Twist: The Bethe Ansatz for Twisted $N = 4$ SYM, JHEP* **08** (2005) 039 [hep-th/0505187]

[10] V. V. Khoze, *Amplitudes in the Beta-Deformed Conformal Yang-Mills, JHEP* **02** (2006) 040 [hep-th/0512194]

[11] C. Durnford, G. Georgiou and V. V. Khoze, *Instanton Test of Non-Supersymmetric Deformations of the AdS(5) x S**5, JHEP* **09** (2006) 005 [hep-th/0606111]

[12] S. Penati, A. Santambrogio and D. Zanon, *Two-Point Correlators in the Beta-Deformed $N = 4$ SYM at the Next-to-Leading Order, JHEP* **10** (2005) 023 [hep-th/0506150]

[13] A. Mauri, S. Penati, A. Santambrogio and D. Zanon, *Exact Results in Planar $N = 1$ Superconformal Yang-Mills Theory, JHEP* **0511** (2005) 024 [hep-th/0507282]

[14] A. Mauri, S. Penati, M. Pirrone, A. Santambrogio and D. Zanon, *On the Perturbative Chiral Ring for Marginally Deformed $N = 4$ SYM Theories, JHEP* **08** (2006) 072 [hep-th/0605145]

[15] F. Elmetti, A. Mauri, S. Penati and A. Santambrogio, *Conformal Invariance of the Planar Beta-Deformed $N = 4$ SYM theory Requires Beta Real, JHEP* **0701** (2007) 026 [hep-th/0606125]

[16] F. Elmetti, A. Mauri, S. Penati, A. Santambrogio and D. Zanon, *Real Versus Complex Beta-Deformation of the N=4 Planar Super Yang-Mills Theory, JHEP* **0710** (2007) 102 [arXiv:0705.1483]

[17] F. Elmetti, A. Mauri and M. Pirrone, *Conformal Invariance and Finiteness Theorems for non-Planar Beta-Deformed N=4 SYM Theory* [arXiv:0710.4864]

[18] D. I. Kazakov and L. V. Bork, *Conformal Invariance = Finiteness and Beta Deformed N=4 SYM Theory JHEP* **0708** (2007) 071 [arXiv:0706.4245]

[19] D. Bundzik and T. Mansson, *The General Leigh-Strassler Deformation and Integrability, JHEP* **01** (2006) 116 [hep-th/0512093]

[20] T. Mansson *The Leigh-Strassler Deformation and the Quest for Integrability, JHEP* **06** (2007) 010 [hep-th/0703150]

[21] A. Agarwal *Open Spin Chains in Super Yang-Mills at Higher Loops: Some Potential Problems with Integrability, JHEP* **08** (2006) 027 [hep-th/0603067]

[22] A. Agarwal *Aspects of Integrability in N =4 SYM, Mod. Phys. Lett.* **A22** (2007) 2549-2563 [arXiv:0708.2747]

[23] N. Beisert and D. Erkal *Yangian Symmetry of Long-Range gl(N) Integrable Spin Chains* (2007) [arXiv:0711.4813]

[24] N. Beisert, *The Complete One-Loop Dilatation Operator of $N = 4$ Super Yang-Mills Theory, Nucl. Phys.* **B676** (2004) 3 [hep-th/0307015]

[25] N. Beisert, *The SU($2|3$) Dynamic Spin Chain, Nucl. Phys.* **B682** (2004) 487 [hep-th/0310252]

[26] B. I. Zwiebel, *Yangian Symmetry at Two-Loops for the SU($2|1$) Sector of $N = 4$ SYM, J. Phys.* **A40** (2007) 1141 [hep-th/0610283]

[27] N. Beisert and B. I. Zwiebel, *On Symmetry Enhancement in the PSU($1,1|2$) Sector of N=4 SYM, JHEP* **10** (2007) 031 [arXiv:0707.1031]

[28] L. Dolan, C. R. Nappi and E. Witten, *A Relation Between Approaches to Integrability in Superconformal Yang-Mills Theory, JHEP* **10** (2003) 017 [hep-th/0308089]

[29] L. Dolan, C. R. Nappi, E. Witten, *Yangian Symmetry in D = 4 Superconformal Yang-Mills Theory, 3rd International Symposium on Quantum Theory and Symmetries (QTS3), Cincinnati, Ohio, 10-14 Sep 2003.* Published in *Cincinnati 2003, Quantum Theory and Symmetries* 300-315 hep-th/0401243]

[30] L Dolan and C. R. Nappi, *Spin Models and Superconformal Yang-Mills Theory", Nucl. Phys.* **B717** (2005) 36 [hep-th/0411020]

[31] N. Reshetikhin, *Multiparameter Quantum Groups and Twisted Quasitriangular Hopf Algebras, Lett. Math. Phys.* **20** (1990) 331

[32] B. Basu-Mallick and P. Ramadevi, *Construction of Yangian Algebra through a Multideformation Parameter Dependent Rational R matrix, Phys. Lett.* **A211** (1996) 339 [hep-th/9409178]

[33] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, Commun. Math. Phys. **252**, 189 (2004) [arXiv:hep-th/0312171].

[34] N. Berkovits, *An alternative string theory in twistor space for N = 4 super-Yang-Mills*, Phys. Rev. Lett. **93**, 011601 (2004) [arXiv:hep-th/0402045].

[35] N. Berkovits and E. Witten, *Conformal supergravity in twistor-string theory*, JHEP **0408**, 009 (2004) [arXiv:hep-th/0406051].

[36] E. Bergshoeff, M. de Roo and B. de Wit, *Extended conformal supergravity*, Nucl. Phys. B **182**, 173 (1981).

[37] E. S. Fradkin and A. A. Tseytlin, *Conformal supergravity*, Phys. Rept. **119**, 233 (1985).

[38] E. S. Fradkin and A. A. Tseytlin, *Conformal anomaly in Weyl theory and anomaly free superconformal theories*, Phys. Lett. B **134**, 187 (1984).

[39] H. Romer and P. van Nieuwenhuizen, *Axial anomalies in N=4 conformal supergravity*, Phys. Lett. B **162**, 290 (1985).

[40] C. M. Bender and P. D. Mannheim, *Giving up the ghost*, arXiv:0807.2607 [hep-th].

[41] M. Abou-Zeid, C. M. Hull and L. J. Mason, *Einstein supergravity and new twistor string theories*, Commun. Math. Phys. **282**, 519 (2008) [arXiv:hep-th/0606272].

[42] S. J. Parke and T. R. Taylor, *An amplitude for n gluon scattering*, Phys. Rev. Lett. **56**, 2459 (1986).

[43] M. L. Mangano and S. J. Parke, *Multiparton amplitudes in gauge theories*, Phys. Rept. **200**, 301 (1991) [arXiv:hep-th/0509223].

[44] F. Cachazo, P. Svrcek and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, JHEP **0409**, 006 (2004) [arXiv:hep-th/0403047].

[45] F. A. Berends, W. T. Giele and H. Kuijf, *On relations between multi - gluon and multigraviton scattering*, Phys. Lett. B **211**, 91 (1988).

[46] K. G. Selivanov, *Gravitationally dressed Parke-Taylor amplitudes*, Mod. Phys. Lett. A **12**, 3087 (1997) [arXiv:hep-th/9711111].

[47] Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein and J. S. Rozowsky, *On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences*, Nucl. Phys. B **530**, 401 (1998) [arXiv:hep-th/9802162].

[48] Z. Bern, A. De Freitas and H. L. Wong, *On the coupling of gravitons to matter*, Phys. Rev. Lett. **84**, 3531 (2000) [arXiv:hep-th/9912033].

[49] Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein and J. S. Rozowsky, *On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences*, Nucl. Phys. B **530**, 401 (1998) [arXiv:hep-th/9802162].

[50] N. E. J. Bjerrum-Bohr, D. C. Dunbar, H. Ita, W. B. Perkins and K. Risager, *MHV-vertices for gravity amplitudes*, JHEP **0601**, 009 (2006) [arXiv:hep-th/0509016].

[51] L. Dolan and P. Goddard, *Tree and loop amplitudes in open twistor string theory*, JHEP **0706**, 005 (2007) [arXiv:hep-th/0703054].

[52] L. Dolan and P. Goddard, "Current Algebra on the Torus," Commun. Math. Phys. **285**, 219 (2009) [arXiv:0710.3743 [hep-th]].

[53] V. G. Drinfeld, *Quantum Groups, J. Sov. Math.* **41** (1988) 898 [*Zap. Nauchn. Semin.* **155**, (1986) 18]

[54] V. G. Drinfeld, *Hopf Algebras and the Quantum Yang-Baxter Equation, Sov. Math. Dokl.* **32** (1985) 254 [*Dokl. Akad. Nauk Ser. Fiz.* **283**, (1985) 1060]

[55] D. Bernard, *An Introduction to Yangian Symmetries, Int. J. Mod. Phys. B* **7** (1993), 3517 [hep-th/9211133]

[56] N. J. MacKay, *Introduction to Yangian Symmetry in Integrable Field Theory, Int. J. Mod. Phys. A* **20** (2005) 7189 [hep-th/0409183]

[57] S. Ferrara and B. Zumino, *Structure of conformal supergravity*, Nucl. Phys. B **134**, 301 (1978).

[58] N. Berkovits and L. Motl, *Cubic twistorial string field theory*, JHEP **0404**, 056 (2004) [arXiv:hep-th/0403187].

[59] I.B. Frenkel and Y. Zhu, *Vertex Operator Algebras Associated to Representations of Affine and Virasoro Algebras*, Duke Math J. **66**, 123 (1992).

[60] M. B. Green, J. H. Schwarz and L. Brink, *N=4 Yang-Mills and N=8 supergravity as limits of string theories*, Nucl. Phys. B **198**, 474 (1982).

[61] J. Bedford, *On perturbative field theory and twistor string theory*, arXiv:0709.3478 [hep-th].

[62] C. Vergu, *Twistors, strings and supersymmetric gauge theories*, arXiv:0809.1807 [hep-th].

[63] C.h. Ahn, *Comments on MHV tree amplitudes for conformal supergravitons from topological B-model*, JHEP **0507**, 004 (2005) [arXiv:hep-th/0504109].

[64] C. Saemann, "Aspects of twistor geometry and supersymmetric field theories within superstring theory," arXiv:hep-th/0603098.

[65] R. Penrose, "Twistor Quantization And Curved Space-Time," Int. J. Theor. Phys. **1**, 61 (1968).